APPLICATIONS OF HOMOTOPY PERTURBATION TRANSFORM METHOD FOR HEAT EQUATION

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Abstract

In the present research, we propose the Homotopy Perturbation Transform Method (HPTM), an efficient method for solving differential equations that combines the Laplace Transform Method with the Homotopy Perturbation Method (HPM). Solving linear and nonlinear partial differential equations, such as the heat equation, is one of its main applications. Ji-Huan He established the HPM technique in 1999, and M. Omran introduced the Homotopy Perturbation Transform technique (HPTM) in 2012. By applying HPTM, the solution process becomes more organized and controllable, which makes it an effective tool for solving heat equations in a variety of scenarios. The precise findings found in the literature were compared with the solutions. The outcomes demonstrate that the HPTM can effectively generate solutions that are precise, converge more quickly, and use less computer resources.

Keywords: HPTM, Heat equation, exact solution, non-local conditions

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1. Introduction

When expressed as partial differential equations and integral equations, real-world issues in scientific domains including solid state physics, plasma physics, fluid mechanics, chemical kinetics, and mathematical biology are typically linear or nonlinear. Numerous effective and straightforward techniques have been put forth and effectively used to address a wide range of issues within the past 20 years. Numerous approximation techniques have been developed, including the differential transform approach [9–10], the Laplace decomposition method [11–12], the variational iteration technique [3–8], and the Adomian decomposition method [1-2]. He [13–16] initially suggested the homotopy perturbation method (HPM), a modern analytical tool for addressing a variety of linear and nonlinear starting and boundary value problems. It combines the usual homotopy and classical perturbation techniques[17-23]. Some scientists have

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since tweaked it to provide faster convergence, more accurate findings, and less processing [24–26]. The linear and nonlinear partial differential equations can be solved with accurate approximations using the HPM, VIM, and ADM methods. However, this approximation is only acceptable for a limited range because these methods can only satisfy boundary conditions in one dimension. This indicates that most analytical techniques have inherent flaws and require a significant amount of computational work. The Adomain decomposition approach is the most clear way to solve partial differential equations, but its use is limited since it requires the computation of complex Adomain polynomials. Due to the challenges posed by the nonlinear variables, the Laplace transform is completely unable to handle the nonlinear equations. We create a very efficient way to deal with these nonlinearities by combining the Laplace transform method with the homotopy perturbation method to solve these shortcomings. Numerous approaches, such as the Adomain decomposition technique, have been put out lately to address nonlinearities [27]. Moreover, the homotopy perturbation approach is coupled with the variational iteration method [29] and Laplace transform method [28] to provide a powerful tool for resolving a variety of nonlinear issues.

This work's main goal is to suggest a fresh HPM modification in order to address the shortcoming. The answer is given by the proposed HPTM in a quickly convergent series, which might lead to a closed form solution. This method's benefit is its ability to combine two effective techniques for finding nonlinear equations' precise solutions. It is important to note that the HPTM is implemented without the use of discretization, constrictive transformations or assumptions, or round-off mistakes. One or two iteration stages also yield extremely precise results across a large range. Unlike the separation of variables technique, which requires initial or boundary conditions, the HPTM approach just requires the starting conditions to provide an analytical solution. Only the findings produced can be justified using the boundary conditions. The suggested approach functions effectively, and the preliminary findings are reassuring and trustworthy. It is important to note that the HPTM can be regarded as a major improvement over earlier techniques and as a substitute for more recent approaches like the Homotopy perturbation method, Variational iteration method, and Adomain's decomposition method. The effectiveness and dependability of the homotopy perturbation transform approach are demonstrated using a number of instances.

The efficiency of the Homotopy Perturbation Transform Method (HPTM) in solving heat equations with starting and boundary conditions has been examined in the current work. An alternative set of numerical examples is provided in the section under "implementation of the method." Lastly, they provide a few closing observations.

2. Analysis of Homotopy Perturbation Transform Method

To illustrate the basic concept of this method, we consider the differential equation:

$$D\,u(x,t) + R\,u(x,t) + Nu(x,t) = g(x,t) \tag{2.1}$$

with the initial conditions:

$$u(x,0) = h(x), \quad u_t(x,0) = f(x)$$

where $D = \frac{\partial}{\partial t}$ is the first order linear differential operator, R is the differential operator of less order than D, N is the general nonlinear differential operator and $g(x,t)$ is the source term. Taking the Laplace transform L on both sides of equation (2.1), we have

$$L[Du(x,t)_t] + L[u(x,t)_{xx}] + L[Nu(x,t)] = L[g(x,t)] \qquad (2.2)$$

Using the differentiation property of the Laplace transform, we obtained

$$L[u(x,t)] = \frac{f(x)}{s} + \frac{1}{s}L[g(x,t)] - \frac{1}{s}L[u(x,t)_{xx}] - \frac{1}{s}L[Nu(x,t)] \quad (2.3)$$

Operating with the inverse Laplace transform on both sides of equation(2.3), we have

$$u(x,t) = G(x,t) - L^{-1}\left[\frac{1}{s}L[u(x,t)_{xx} + Nu(x,t)]\right] \qquad (2.4)$$

where $G(x,t)$ is the term arising from the source term and the prescribed initial conditions. Now, applying the Homotopy perturbation method(HPM), we have,

$$u(x,t) = \sum_{n=0}^{\infty} p^n u_n(x,t) \qquad (2.5)$$

and the nonlinear term can be decomposed as

$$N u(x,t) = \sum_{n=0}^{\infty} p^n H_n(u) \qquad (2.6)$$

For some He's polynomial $H_n(u)$ that are given by

$$H_n(u_0, u_1, u_2, \ldots, u_n) = \frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty} p^i u_i\right)\right]_{p=0}, \; n = 0, 1, 2, 3.... \quad (2.7)$$

Substituting equations (2.5) and (2.6) in Equation (2.4), we have

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = G(x,t) - p\left(L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} p^n u_n(x,t)_{xx} + \sum_{n=0}^{\infty} p^n H_n(u)]\right]\right) \; (2.8)$$

Which is the coupling of the Laplace transform and the HPM using He's polynomials.

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Comparing the coefficient of like powers of p, the following approximation are obtained.

$$p^0 : \ u_0(x,t) = G(x,t)$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s}L[u_0(x,t)_{xx} + H_0(u)]\right]$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s}L[u_1(x,t)_{xx} + H_1(u)]\right] \qquad (2.9)$$

$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s}L[u_2(x,t)_{xx} + H_2(u)]\right]$$

and so on. Hence the approximate solution of Eq.(2.1) is given by

$$u(x,t) = \lim_{p\to 1} u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + ...$$

3. Homotopy Perturbation Method and He's polynomial

The homotopy perturbation method is a technique for solving functional equations of various kinds in the form:

$$u - N(u) = f \qquad (2.10)$$

Where N is nonlinear operator from Hilbert space H to H, u is unknown function and f is known function in H.

Consider Eq.(2.10) in the form

$$L(u) = u - f(x) - N(u) \qquad (2.11)$$

with solution $u(x)$. As possible remedy, we can define homotopy $H(u, p)$ as follows:

$$H(u,\ 0) = F(u), \quad H(u,\ 1) = L(u)$$

Where $F(p)$ is an integral operator with known solution u_0 which can be obtained easily, typically we may choose a convex homotopy in the form

$$H(u, p) = (1 - p)F(u) + pL(u) \qquad (2.12)$$

And continuously trace implicitly defined curve from starting point $H(u_0, 0) = F(u),$ to the solution function $H(u,\ 1) = F(u),$ the embedding parameter p monotonically increase from zero to unit as the trivial problem $F(u)=0$ is continuously deformed form to original problem $L(u)=0$, the embedding parameter, $p\in[0,\,1]$ can be considered as an expanding parameter

$$u=u_0+pu_1+p^2u_2+.... \qquad (2.13)$$

When $p\to 1$, Eq.(2.12) corresponds to Equations(2.11) and (2.13) becomes the approximate of Eq.(2.11).

i. e., $u=\lim_{p\to 1} u=u_0+u_1+u_2+....$ (2.14)

4. Numerical Applications

i) Linear Schrodinger equation

Consider the linear Schrödinger equation from A. M. Wazwaz [30]

$u_t+iu_{xx}=0$ (2.15)

with the initial condition

$$u(x,0)=1+\cosh(2x) \qquad (2.16)$$

Where $u(x,t)$ is a complex function and $i^2=-1$.

Solution: Taking the Laplace transform on both sides of Eq. (2.15) subject to the initial condition (2.16), we have

$$L[u(x,t)]=\frac{1+\cosh(2x)}{s}-\frac{1}{s}i\,L[u_{xx}] \qquad (2.17)$$

The inverse Laplace transform of Eq.(2.17) is

$$u(x,t)=1+\cosh(2x)-L^{-1}\left[\frac{1}{s}i\,L[u_{xx}]\right] \qquad (2.18)$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty}p^n u_n(x,t)=1+\cosh(2x)-p\left(L^{-1}\left[\frac{1}{s}i\,L[(\sum_{n=0}^{\infty}p^n u_n(x,t))_{xx}]\right]\right) \qquad (2.19)$$

Comparing the coefficients of same powers of p, we have

$$p^0: u_0(x,t)=1+\cosh(2x)$$

$$p^1: u_1(x,t)=-L^{-1}\left[\frac{1}{s}iL[(u_0(x,t)_{xx}\right]=(-4it)\cosh(2x)$$

$$p^2: u_2(x,t)=-L^{-1}\left[\frac{1}{s}iL[(u_1(x,t)_{xx}\right]=\frac{(-4it)^2\cosh(2x)}{2!}$$

$$p^3: u_3(x,t)=-L^{-1}\left[\frac{1}{s}iL[(u_2(x,t)_{xx}\right]=\frac{(-4it)^3\cosh(2x)}{3!}$$

and so on. Therefore, the solution $u(x,t)$ is given by

$$u(x,t) = \lim_{p\to 1} u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= 1 + \cosh(2x) + (-4it)\cosh(2x) + \frac{(-4it)^2 \cosh(2x)}{2!} + \frac{(-4it)^3 \cosh(2x)}{3!} + \ldots$$

$$= 1 + e^{-4it} \cosh(2x)$$

which is the exact solution.

ii) Consider linear homogeneous diffusion equation

$$u_t = u_{xx} - u, \quad 0 < x < 1,\ t > 0 \qquad (2.20)$$

with the initial condition

$$u(x,0) = \sin(\pi x), \quad 0 < x < 1 \qquad (2.21)$$

and boundary conditions are given by

$$u(0,t) = u(1,t) = 0, \quad t > 0 \qquad (2.22)$$

Where $u(x,t)$ is a complex function and $i^2 = -1$.

Solution: Taking the Laplace transform on both sides of Eq.(2.20) subject to the initial condition (2.21), we have

$$L[u(x,t)] = \frac{\sin(\pi x)}{s} - \frac{1}{s} L[u - u_{xx}] \qquad (2.23)$$

The inverse Laplace transform of Eq.(2.22) is

$$u(x,t) = \sin(\pi x) - L^{-1}\left[\frac{1}{s} L[u - u_{xx}]\right] \qquad (2.24)$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sin(\pi x) - p\left(L^{-1}\left[\frac{1}{s} L[(\sum_{n=0}^{\infty} p^n \left(u_n(x,t) - u_n(x,t)_{xx}\right)]\right]\right) \qquad (2.25)$$

Comparing the coefficients of same powers of p, we have

$$p^0 : u_0(x,t) = \sin(\pi x)$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s} L[(u_0(x,t) - u_0(x,t)_{xx}]\right] = -(\pi^2 + 1)t\sin(\pi x)$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s} L[(u_1(x,t) - u_1(x,t)_{xx}]\right] = \frac{(\pi^2 + 1)^2 t^2}{2!}\sin(\pi x)$$

$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s} L[(u_2(x,t) - u_2(x,t)_{xx}]\right] = -\frac{(\pi^2 + 1)^3 t^3}{3!}\sin(\pi x)$$

$$p^4 : u_4(x,t) = -L^{-1}\left[\frac{1}{s} L[(u_3(x,t) - u_3(x,t)_{xx}]\right] = \frac{(\pi^2 + 1)^4 t^4}{4!}\sin(\pi x)$$

and so on. Therefore, the solution $u(x,t)$ is given by

$$u(x,t) = \lim_{p\to 1} u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= \sin(\pi x)\left[1 - (\pi^2+1)t + \frac{(\pi^2+1)^2 t^2}{2!} - \frac{(\pi^2+1)^3 t^3}{3!} + \frac{(\pi^2+1)^4 t^4}{4!} - \ldots\right]$$ Thi

$$= \sin(\pi x)\, e^{-(\pi^2+1)t}$$

s the exact solution.

iii) Consider non-homogeneous diffusion equation

$$u_t - 3u_{xx} = x, \quad 0 < x < \pi,\ t > 0 \tag{2.26}$$

with the initial condition

$$u(x,0) = \sin\ x,\ \ 0 < x < \pi \tag{2.27}$$

and boundary conditions are given by

$$u(0,t) = 0,\ \ u(\pi,t) = \pi t, \quad t > 0 \tag{2.28}$$

Solution: Taking the Laplace transform on both sides of Eq.(2.26) subject to the initial condition (2.27), we have

$$L[u(x,t)] = \frac{\sin(x)}{s} - \frac{1}{s}\, L[-3u_{xx} - x] \tag{2.29}$$

The inverse Laplace transform of Eq.(2.29) is

$$u(x,t) = \sin(x) - L^{-1}\left[\frac{1}{s}\, L[-3u_{xx} - x]\right] \tag{2.30}$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = \sin(x) - p\left(L^{-1}\left[\frac{1}{s} L[\sum_{n=0}^{\infty} p^n\left(-3u_n(x,t)_{xx} - x\right)]\right]\right) \tag{2.31}$$

Comparing the coefficients of same powers of p, we have

$$p^0 : u_0(x,t) = \sin(x)$$

$$p^1 : u_1(x,t) = -L^{-1}\left[\frac{1}{s} L[-3u_0(x,t)_{xx} - x]\right] = -3t\sin(x) + xt$$

$$p^2 : u_2(x,t) = -L^{-1}\left[\frac{1}{s} L[-3u_1(x,t)_{xx} - x]\right] = \frac{9t^2}{2!}\sin(x)$$

$$p^3 : u_3(x,t) = -L^{-1}\left[\frac{1}{s} L[-3u_2(x,t)_{xx} - x]\right] = -\frac{27t^3}{3!}\sin(x)$$

$$p^4 : u_4(x,t) = -L^{-1}\left[\frac{1}{s} L[-3u_3(x,t)_{xx} - x]\right] = \frac{81t^3}{4!}\sin(x)$$

and so on. Therefore, the solution $u(x,t)$ is given by

$$u(x,t) = \lim_{p\to 1} u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \dots$$

$$= \sin(x)\left[1 - 3t + \frac{9t^2}{2!} - \frac{27t^3}{3!} + \frac{81t^2}{4!} - \dots\right] + xt$$

$$= \sin(\pi x)\, e^{-3t} + xt$$

which is the required exact solution available in the literature.

iv) Consider Heat equation

$$u_t = u_{xx}, \quad 0 < x < 1,\ t > 0 \qquad (2.32)$$

with the initial condition

$$u(x,0) = 3\sin(2\pi x), \ 0 < x < 1 \qquad (2.33)$$

and boundary conditions are given by

$$u(0,t) = u(1,t) = 0, \quad t > 0 \qquad (2.34)$$

Solution: Taking the Laplace transform on both sides of Eq.(2.32) subject to the initial condition (2.33), we have

$$L[u(x,t)] = \frac{3\sin(2\pi x)}{s} + \frac{1}{s}L[u_{xx}] \qquad (2.35)$$

The inverse Laplace transform of Eq.(2.35) is

$$u(x,t) = 3\sin(2\pi x) + L^{-1}\left[\frac{1}{s}L[u_{xx}]\right] \qquad (2.36)$$

Now applying the HPM, we get

$$\sum_{n=0}^{\infty} p^n u_n(x,t) = 3\sin(2\pi x) + p\left(L^{-1}\left[\frac{1}{s}L[\sum_{n=0}^{\infty} p^n u_n(x,t)_{xx}]\right]\right) \qquad (2.37)$$

Comparing the coefficients of same powers of p, we have

$$p^0 : u_0(x,t) = 3\sin(2\pi x)$$

$$p^1 : u_1(x,t) = L^{-1}\left[\frac{1}{s}L[u_0(x,t)_{xx}]\right] = -3(4\pi^2)t\sin(2\pi x)$$

$$p^2 : u_2(x,t) = L^{-1}\left[\frac{1}{s}L[u_1(x,t)_{xx}]\right] = 3(4\pi^2)^2\frac{t^2}{2!}\sin(2\pi x)$$

$$p^3 : u_3(x,t) = L^{-1}\left[\frac{1}{s}L[u_2(x,t)_{xx}]\right] = -3(4\pi^2)^3\frac{t^3}{3!}\sin(2\pi x)$$

$$p^4 : u_4(x,t) = L^{-1}\left[\frac{1}{s}L[u_3(x,t)_{xx}]\right] = 3(4\pi^2)^4\frac{t^3}{4!}\sin(2\pi x)$$

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and so on. Therefore, the solution $u(x,t)$ is given by

$$u(x,t) = \lim_{p\to 1} u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + u_3(x,t) + \ldots$$

$$= 3\sin(2\pi x)\left[1 - (4\pi^2)t + \frac{(4\pi^2)^2 t^2}{2!} - \frac{(4\pi^2)^3 t^3}{3!} + \frac{(4\pi^2)^4 t^2}{4!} - \ldots\right] + xt$$

$$= 3\sin(2\pi x)\, e^{-4\pi^2 t}$$

which is the required exact solution available in the literature.

3. Conclusion

Using He's polynomials, the homotopy perturbation transform technique (HPTM) has a straightforward solution process. When compared to the traditional homotopy perturbation approach, it can reduce the amount of computing work. Furthermore, no random beginning estimate is needed. In conclusion, we found that the Homotopy perturbation transform approach fully utilizes all other available techniques.

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