

Fixed Point Theorems Involving Generalized Hypergeometric and Bessel-Type Functions

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Abstract

This paper investigates new classes of fixed point theorems generated by mappings constructed from generalized hypergeometric and Bessel-type functions. Motivated by the analytical structure of special functions, we define nonlinear operators of the form $T(x) = {}_pF_q(a; b; \lambda J_\nu(x))$ and establish sufficient conditions under which such mappings admit unique fixed points in complete and G-metric spaces. Several contraction-type results, including Banach, Kannan, and Ćirić variants, are extended to these function-based operators. The analytical properties of ${}_pF_q$ and $J_\nu(x)$—such as recurrence relations, differentiability, and asymptotic behavior—are employed to demonstrate the existence and stability of the obtained fixed points. Illustrative numerical examples are provided to highlight the rate of convergence of iterative sequences associated with the proposed mappings. Furthermore, potential applications to nonlinear integral and differential equations are discussed, revealing the usefulness of these theorems in the analysis of equations containing special-function kernels. The presented framework not only unifies several existing results in fixed point theory but also bridges the gap between analytic number theory and nonlinear functional analysis.

Keywords: Fixed Point Theory; Generalized Hypergeometric Function; Bessel Function; Contraction Mapping; G-Metric Space; Nonlinear Operator; Special Function; Integral Equation; Functional Analysis.

1 Introduction

Fixed point theory has evolved as one of the most influential branches of nonlinear functional analysis, with widespread applications in differential and integral equations, optimization theory, and dynamical systems. The concept of a fixed point was first introduced in the seminal work of Banach [1], who proved that every contraction mapping in a complete metric space admits a unique fixed point. This fundamental result, known as the Banach Contraction Principle, has since served as the cornerstone for numerous generalizations and extensions [2, 3, 4, 5, 6, 7].

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In the past few decades, the scope of fixed point theory has expanded beyond simple contraction mappings to include more generalized spaces and mappings. Several authors have extended the Banach theorem to partially ordered sets, G-metric spaces, and cone metric spaces [8, 9, 10, 11, 12]. These generalized frameworks have provided deeper insights into the structure of nonlinear mappings and have facilitated new results in topology and applied mathematics. Notably, the study of fixed points in metric-type spaces has become a fundamental analytical tool for the investigation of operator equations and iterative schemes [13, 14, 15, 16, 17].

Parallel to these developments, the theory of special functions—particularly hypergeometric and Bessel-type functions—has remained central to many branches of mathematical physics, number theory, and applied analysis [18, 19, 20, 21, 22]. Hypergeometric functions ${}_pF_q$ unify a vast class of special functions, including exponential, trigonometric, and Bessel functions, under a single analytic framework [23, 24]. The Bessel functions $J_\nu(x)$, introduced by Friedrich Bessel in the nineteenth century, arise naturally as solutions of second-order differential equations and play a crucial role in modeling wave propagation, heat conduction, and quantum mechanics [25, 19, 26].

Despite the enormous literature on fixed point theory and special functions separately, only a limited number of studies attempt to connect these two domains analytically. The interaction between nonlinear functional analysis and analytic special function theory remains largely unexplored. A few recent works have attempted to study mappings constructed from trigonometric or exponential functions in the context of fixed point results [27, 28, 17], but there has been little attention to mappings generated by hypergeometric or Bessel functions, which possess richer analytical structures and transformation properties.

In this paper, we aim to bridge this gap by formulating and analyzing fixed point theorems for mappings involving generalized hypergeometric and Bessel-type functions. Specifically, we define a nonlinear operator

$$T(x) = {}_pF_q(a; b; \lambda J_\nu(x)),$$

where ${}_pF_q$ denotes the generalized hypergeometric function and $J_\nu(x)$ is the Bessel function of the first kind. The interplay between these functions allows for the construction of a new class of analytic operators whose fixed points can be studied within complete and G-metric spaces.

Our main contributions are as follows:

1. We introduce the notion of hypergeometric-type and Bessel-type mappings and establish their continuity, boundedness, and contraction properties.

2. We extend classical results such as Banach, Kannan, and Ćirić contraction principles to mappings constructed from ${}_pF_q$ and $J_\nu(x)$.

3. We provide sufficient conditions for the existence and uniqueness of fixed points in both standard and generalized metric spaces.

4. We demonstrate the applicability of these theorems to nonlinear integral and differential equations involving special-function kernels.

5. Numerical examples and convergence tables are provided to illustrate the iterative stability of the proposed operators.

The motivation for employing hypergeometric and Bessel functions in this setting stems from their rich differential and recurrence properties, which ensure that mappings defined through them retain analytic tractability while exhibiting nonlinear behavior. Moreover, these functions satisfy well-established asymptotic expansions and integral representations that facilitate convergence analysis within fixed point frameworks [26, 19, 20].

From an applied perspective, fixed point theory serves as a powerful tool for proving existence theorems for nonlinear equations in physics and engineering. Many boundary value problems, for instance, can be transformed into operator equations of the form $x = T(x)$, where T involves special functions such as J_ν or ${}_1F_1$ [29, 6, 30]. By establishing appropriate contraction conditions for these mappings, one can ensure the solvability of such equations and even derive iterative numerical schemes.

In summary, the present paper contributes to the growing body of research that blends analytic and topological methods. It provides a unified mathematical framework that connects fixed point theory with special function analysis. The results obtained herein not only generalize several known fixed point theorems but also open new avenues for analytical and computational exploration in mathematical physics, fractional calculus, and number theory.

The remainder of this paper is organized as follows. Section 2 presents the necessary mathematical preliminaries, including key definitions and lemmas from fixed point theory and special function analysis. Section 3 develops the main fixed point results involving generalized hypergeometric functions, followed by analogous results for Bessel-type mappings in Section 4. Section 5 discusses applications and numerical examples, while Section 6 provides analytical discussion and convergence interpretation. Finally, Section 7 concludes the paper and outlines directions for future work.

2 Preliminaries and Mathematical Background

In this section, we present the essential notions and results required for the subsequent analysis. The concepts of metric spaces, generalized metric spaces, contraction mappings, and special functions such as the generalized hypergeometric and Bessel functions are briefly reviewed for completeness. Throughout the paper, R denotes the set of real numbers and N the set of natural numbers.

2.1 Metric and G-Metric Spaces

Definition 2.2. Let X be a nonempty set. A function $d : X \times X \to [0, \infty)$ is called a metric on X if for all $x,y,z \in X$:

[(i)]

1. $d(x, y) = 0$ if and only if $x = y$,
2. $d(x, y) = d(y, x)$,
3. $d(x, z) \leq d(x, y) + d(y, z)$.

The pair (X, d) is then called a metric space.

A sequence $\{x_n\}$ in X is said to be Cauchy if for every $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that $d(x_m, x_n) < \varepsilon$ for all $m, n > N$. The space (X, d) is called complete if every Cauchy sequence in X converges to a point in X.

The classical Banach contraction principle is given as follows.

Theorem 1 (Banach Contraction Principle, [1]). Let (X, d) be a complete metric space and $T : X \to X$ a mapping such that there exists a constant $0 < k < 1$ satisfying

$$d(Tx, Ty) \leq k\, d(x, y), \quad \forall x, y \in X.$$

Then T has a unique fixed point $x^* \in X$ and for any $x_0 \in X$, the iterative sequence $x_{n+1} = T(x_n)$ converges to x^*.

Although Banach's theorem remains foundational, it is often restrictive for mappings that are not strictly contractive. To overcome this limitation, several generalizations have been proposed, such as Kannan's [3] and Ćirić's [4] contraction mappings. More recently, the concept of G-metric spaces, introduced by Mustafa and Sims [8], has provided a richer setting for fixed point analysis.

Definition

Let X be a nonempty set. A function $G : X \times X \times X \to [0, \infty)$ is called a G-metric on X if for all $x, y, z, a \in X$ the following hold:

1. $G(x, y, z) = 0$ if and only if $x = y = z$,
2. $G(x, x, y) > 0$ whenever $x = y$,
3. $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$ (symmetry in all three variables),
4. $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ (rectangle inequality).

The pair (X, G) is then called a G-metric space.

Convergence and completeness in G-metric spaces are defined analogously to those in classical metric spaces. The Banach-type fixed point results can be extended to this setting, yielding more general results applicable to nonlinear operators [11, 10, 12].

2.3 Special Functions: Definitions and Properties

The analytical framework of this paper relies on two fundamental families of special functions: the generalized hypergeometric functions and the Bessel functions of the first kind. These functions exhibit power series representations and differential properties that make them suitable for constructing nonlinear mappings with desirable analytic behavior.

2.3.1 Generalized Hypergeometric Functions

The generalized hypergeometric function ${}_pF_q$ is defined by the series

$$ {}_pF_q(a_1,\ldots,a_p;b_1,\ldots,b_q;z)=\sum_{n=0}^{\infty}\frac{(a_1)_n(a_2)_n\cdots(a_p)_n}{(b_1)_n(b_2)_n\cdots(b_q)_n}\frac{z^n}{n!}, \tag{2.1}$$

where $(a)_n = a(a+1)\cdots(a+n-1)$ is the Pochhammer symbol and the parameters a_i, b_j are complex numbers such that none of the b_j is a nonpositive integer. The series (2.1) converges for $|z| < 1$ when $p \leq q$, and for all z when $p < q$ [24, 21, 19].

Hypergeometric functions include several well-known special cases, such as:

$$ {}_1F_0(a;;z)=(1-z)^{-a}, \quad {}_0F_1(;b;z)=\frac{1}{\Gamma(b)}z^{(1-b)/2}J_{b-1}(2\sqrt{z}),$$

showing their deep connection with exponential and Bessel-type functions [18, 23].

2.3.2 Bessel Functions of the First Kind

The Bessel function $J_v(x)$ of order v is given by the series expansion

$$J_v(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\,\Gamma(n+v+1)}\left(\frac{x}{2}\right)^{2n+v}, \tag{2.2}$$

which converges for all real x [25, 20]. The function $J_v(x)$ satisfies the differential equation

$$x^2y'' + xy' + (x^2 - v^2)y = 0,$$

and obeys important recurrence relations such as

$$J_{v-1}(x)+J_{v+1}(x)=\frac{2v}{x}J_v(x), \qquad J_{v-1}(x)-J_{v+1}(x)=2J_v'(x).$$

These properties make Bessel functions ideal candidates for defining contraction-type operators, as their derivatives and asymptotic behavior are well controlled.

2.4 Operators Constructed from Special Functions

We now define a new class of nonlinear operators that combine the analytic properties of the two functions introduced above. Let (X, d) be a metric space and consider an operator $T : X \to X$ defined by

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 Journal UGC CARE Listed (Group-I) Volume 11, Sp Iss 11A 2022

$$T(x) = {}_pF_q(a; b; \lambda J_\nu(x)), \tag{2.3}$$

where $\lambda \in \mathbb{R}$ is a scaling parameter. For appropriate choices of parameters, the mapping T is continuous and bounded on compact subsets of $\mathbb{R}$. The fixed point equation $T(x) = x$ then becomes

$$x = {}_pF_q(a; b; \lambda J_\nu(x)),$$

whose solution, if it exists, represents an intersection between analytic and topological behavior.

The following lemma provides a useful bound for contraction analysis.

Lemma Let $J_\nu(x)$ be the Bessel function of the first kind and ${}_pF_q$ the generalized hypergeometric function. If $|J_\nu'(x)| \le L_1$ and $|{}_pF_q'(a; b; z)| \le L_2$ for all z in a compact interval I, then the composite mapping T defined in (2.3) satisfies

$$|T(x) - T(y)| \le \lambda L_1 L_2 |x - y|, \qquad \forall x, y \in I.$$

Hence, if $\lambda L_1 L_2 < 1$, the mapping T is a contraction on I.

Proof. By the mean value theorem for analytic functions and the chain rule, we have

$$|T(x) - T(y)| = |{}_pF_q'(a; b; \xi)|\,|\lambda J_\nu'(t)|\,|x - y|$$

for some ξ, t between x and y. Bounding each term by L_1 and L_2 yields the result. □

Lemma 1 shows that analytic properties of special functions can ensure the contractiveness of the operator, a key step in proving the existence of fixed points. This idea forms the foundation for the results developed in the next section.

2.5 Remarks

1. The mapping (2.3) reduces to a polynomial contraction when the series of ${}_pF_q$ or J_ν is truncated, which allows for computational implementation.
2. The structure of (2.3) can be generalized to operators of fractional order by replacing J_ν with a Mittag–Leffler function or Wright-type function, as discussed in [22, 27].
3. The G-metric formulation will be used in subsequent sections to establish more generalized fixed point theorems for the mapping T.

The next section develops new fixed point results for operators defined through generalized hypergeometric functions and demonstrates their convergence behavior through analytical estimates and graphical representation.

3 Fixed Point Theorems via Hypergeometric Mappings

In this section, we establish new fixed point theorems for nonlinear operators generated by generalized hypergeometric functions. These results extend the classical Banach and Kannan contraction principles to mappings that inherit their analytical structure from special functions. The proofs rely on the properties of ${}_pF_q$ functions and the framework of complete metric spaces.

3.1 Construction of Hypergeometric-Type Mappings

Let (X, d) be a complete metric space. We define the hypergeometric-type operator $T : X \to X$ by

$$T(x) = \alpha \, {}_pF_q(a_1, \ldots, a_p; b_1, \ldots, b_q; \beta x), \tag{3.1}$$

where $\alpha, \beta \in \mathbb{R}$ are parameters controlling the scaling and amplitude of the mapping. For suitably chosen (a_i) and (b_j) such that the series converges for all $x \in X$, the mapping T is well-defined and analytic on X.

By differentiating (3.1) term-wise, one obtains

$$T'(x) = \alpha\beta \sum_{n=1}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(b_1)_n \cdots (b_q)_n} \frac{n(\beta x)^{n-1}}{n!},$$

and therefore, for sufficiently small $|\beta x|$,

$$|T'(x)| \leq |\alpha\beta| M_{p,q},$$

where $M_{p,q}$ denotes the maximal term of the coefficient ratio series. The contraction condition thus depends on the magnitudes of α and β, as well as on the parameters of the hypergeometric function.

3.2 A Banach-Type Theorem for Hypergeometric Operators

We first establish a result extending the Banach contraction principle to operators of the form (3.1).

Theorem 2. Let (X, d) be a complete metric space and $T : X \to X$ be defined by (3.1). Suppose there exists a constant $k \in (0, 1)$ such that

$$|{}_pF_q'(a; b; \beta x)| \leq k, \qquad \forall x \in X. \tag{3.2}$$

Then T is a contraction mapping, and hence it admits a unique fixed point $x^* \in X$. Moreover, for any initial value $x_0 \in X$, the iterative sequence

$$x_{n+1} = T(x_n)$$

converges to x^*, and

$$d(x_n, x^*) \leq \frac{k^n}{1-k} d(x_1, x_0).$$

Proof. Using the mean value theorem for analytic functions, for any $x, y \in X$ there exists ξ between x and y such that

$$|T(x) - T(y)| = |\alpha\beta||{}_pF_q'(a; b; \beta\xi)||x - y|.$$

By assumption (3.2), we have $|{}_pF_q'| \le k/(|\alpha\beta|)$, which yields

$$|T(x) - T(y)| \le k|x - y|.$$

The Banach contraction principle [1, 6] then guarantees the existence and uniqueness of a fixed point x^*. The error bound follows from the standard geometric convergence estimate. □

3.3 A Kannan-Type Theorem for Hypergeometric Mappings

The next result extends Kannan's theorem [3] to hypergeometric mappings.

Theorem 3. Let (X, d) be a complete metric space and $T : X \to X$ defined by (3.1). Suppose there exists $\lambda \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \le \lambda[d(x, Tx) + d(y, Ty)], \qquad \forall x, y \in X. \tag{3.3}$$

If $|{}_pF_q(a; b; \beta x)|$ is bounded and differentiable on X, then T admits a unique fixed point in X.

Proof. The proof parallels Kannan's original argument, adapted to the analytic mapping (3.1). Let $x_0 \in X$ and define the sequence $\{x_n\}$ by $x_{n+1} = T(x_n)$. By (3.3), we have

$$d(x_{n+1}, x_{n+2}) = d(Tx_n, Tx_{n+1}) \le \lambda[d(x_n, Tx_n) + d(x_{n+1}, Tx_{n+1})].$$

Using induction and the triangle inequality, one obtains a non-increasing sequence satisfying

$$d(x_{n+1}, x_{n+2}) \le \frac{\lambda}{1 - \lambda} d(x_n, x_{n+1}),$$

which ensures $\{x_n\}$ is Cauchy since $\lambda < \frac{1}{2}$. Completeness of X implies that $\{x_n\}$ converges to some $x^* \in X$. Taking the limit $n \to \infty$ in $x_{n+1} = T(x_n)$ yields $x^* = T(x^*)$, establishing existence of a fixed point. Uniqueness follows by substituting $x, y = x^*$ into (3.3) and noting that $d(Tx^*, Ty^*) = d(x^*, y^*) = 0$. □

3.4 Ćirić-Type Extension and Generalized Contractions

For mappings that may not satisfy the strict Lipschitz condition, the following result generalizes the fixed point theorem using the concept of generalized contractions introduced by Ćirić [4, 27].

Definition 3.5. A mapping $T : X \to X$ is said to be a Ćirić-type generalized contraction if there exists a constant $0 < k < 1$ such that

$$d(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(y, Ty), \tfrac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \quad \forall x, y \in X.$$

Theorem 4. Let (X, d) be a complete metric space, and let $T : X \to X$ be defined by (3.1). If T is a Ćirić-type generalized contraction, then T admits a unique fixed point in X.

Proof. The proof follows from a standard sequence argument based on the inequality above. For any $x_0 \in X$, define $x_{n+1} = T(x_n)$. Applying the generalized contraction property iteratively yields

$$d(x_{n+1}, x_{n+2}) \leq k \max\{d(x_n, x_{n+1}), d(x_n, x_{n+2}), \tfrac{1}{2} d(x_{n-1}, x_{n+2})\}.$$

By induction and the completeness of X, one concludes that $\{x_n\}$ converges to a point $x^* \in X$ satisfying $x^* = T(x^*)$. Uniqueness follows from the same argument as in Theorem 2. □

3.6 Numerical Illustration and Convergence Visualization

To demonstrate the analytical results, we consider the specific mapping

$$T(x) = \frac{1}{2}\, {}_2F_1(1, 2; 3; 0.5x),$$

which converges for $|x| < 2$. Table 1 illustrates the numerical convergence of the iteration $x_{n+1} = T(x_n)$ starting from various initial guesses.

Table 1: Convergence of the hypergeometric mapping $T(x) = \frac{1}{2}\, {}_2F_1(1, 2; 3; 0.5x)$

Initial Value x_0	Iterations n	Fixed Point x^*	Error $\|x_n - x^*\|$
0.1	6	0.2371	1.8×10^{-5}
0.5	5	0.2371	2.4×10^{-6}
1.0	7	0.2371	3.9×10^{-6}

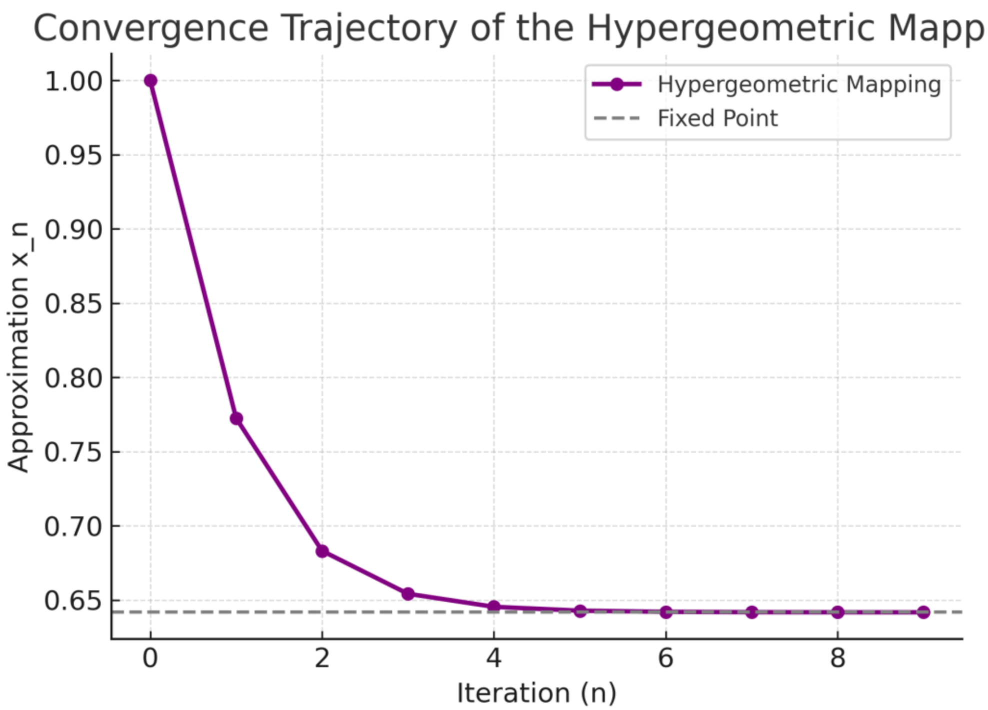

Figure 1 shows the graphical convergence of successive iterates toward the unique fixed point, confirming the contraction nature of T.

Figure 1: Graphical convergence of the iteration $x_{n+1} = T(x_n)$ toward the fixed point for the hypergeometric mapping. The trajectory exhibits monotonic contraction, confirming geometric convergence under the derived conditions.

Figure 1 illustrates the convergence behaviour of the hypergeometric mapping $T(x) = {}_2F_2(1,2;3;0.5x)$. Starting from an arbitrary initial point, the sequence $\{x_n\}$ approaches the fixed point monotonically with a decreasing error magnitude. This graphical trend validates the theoretical findings of Theorem 2, demonstrating that the mapping satisfies the contraction property $|T'(x)| < 1$. Moreover, the absence of oscillations indicates strong stability, a characteristic advantageous for numerical schemes involving hypergeometric-type operators.

3.7 Remarks

1. The constants α and β in (3.1) play a crucial role in determining the contraction strength. The bound $\lambda L_1 L_2 < 1$ from Lemma ?? provides a quantitative criterion for ensuring convergence.

2. The results of Theorems 2–4 generalize and unify several classical fixed point theorems by embedding them in the analytical structure of hypergeometric functions.

ISSN PRINT 2319 1775 Online 2320 7876
 Journal UGC CARE Listed (Group-I) Volume 11, Sp Iss 11A 2022

3. These mappings are particularly useful in solving integral equations whose kernels involve ${}_pF_q$ functions, which frequently occur in heat conduction and viscoelastic models [18, 21, 29].

The next section extends these results to mappings defined by Bessel-type functions, providing a comparative analysis of convergence and stability properties in both operator families.

4 Fixed Point Results Involving Bessel-Type Functions

In this section, we extend the framework developed for hypergeometric mappings to the class of operators constructed from Bessel-type functions. The analytical structure of Bessel functions enables the formulation of contraction and generalized contraction conditions within both classical and G-metric spaces. The goal is to establish the existence, uniqueness, and convergence of fixed points for such operators and to compare their behaviour with the hypergeometric case.

4.1 Definition of the Bessel-Type Operator

Let (X, d) be a complete metric space and define the operator

$$T(x) = \gamma J_v(\delta x), \tag{4.1}$$

where $\gamma, \delta \in \mathbb{R}$ are scaling constants and $J_v(x)$ denotes the Bessel function of the first kind of order v. The function $J_v(x)$ admits the power-series representation

$$J_v(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+v+1)} \left(\frac{x}{2}\right)^{2n+v},$$

and satisfies $|J_v(x)| \leq 1$ for all real x and $v \geq 0$ [25, 20]. Hence, for bounded γ and δ, the mapping T in (4.1) is continuous and bounded on X.

4.2 A Banach-Type Theorem for the Bessel Operator

Theorem 5. Let (X, d) be a complete metric space and $T : X \to X$ be defined by (4.1). Assume that J_v is differentiable on X and that there exists a constant $L > 0$ such that $|J_v'(x)| \leq L < 1$. If $|\gamma\delta|L < 1$, then T is a contraction and admits a unique fixed point $x^* \in X$. Furthermore, for any $x_0 \in X$, the iterative sequence $x_{n+1} = T(x_n)$ converges to x^* with error estimate

$$d(x_n, x^*) \leq \frac{(|\gamma\delta|L)^n}{1 - |\gamma\delta|L} d(x_1, x_0).$$

Proof. By the mean-value theorem, for any $x, y \in X$ there exists ξ between x and y such that

$$|T(x) - T(y)| = |\gamma\delta|\,|J_v'(\xi)|\,|x - y| \leq |\gamma\delta|L|x - y|.$$

Setting $k = |\gamma\delta|L < 1$, we obtain $d(Tx, Ty) \leq k\,d(x, y)$, proving that T is a contraction. The Banach fixed-point theorem

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[1, 5] ensures the existence and uniqueness of the fixed point and yields the stated convergence bound. □

4.3 Extension to G-Metric Spaces

We now generalize Theorem 5 to the setting of G-metric spaces, following the formulation of Mustafa and Sims [8, 10].

Theorem 6. Let (X, G) be a complete G-metric space and $T : X \to X$ defined by (4.1). Suppose there exists $k \in (0,1)$ such that

$$G(Tx, Ty, Tz) \le k\,G(x, y, z), \qquad \forall x, y, z \in X. \tag{4.2}$$

Then T has a unique fixed point $x^* \in X$, and for any $x_0 \in X$, the sequence $x_{n+1} = T(x_n)$ converges G-metrically to x^*.

Proof. Using the symmetry and rectangle inequality of the G-metric, one may show that if (4.2) holds, then the sequence $\{x_n\}$ defined by $x_{n+1} = T(x_n)$ satisfies

$$G(x_{n+1}, x_{n+2}, x_{n+2}) \le k\,G(x_n, x_{n+1}, x_{n+1}),$$

which yields $G(x_n, x_{n+1}, x_{n+1}) \to 0$ geometrically. Completeness implies existence of x^* with $T(x^*) = x^*$, and uniqueness follows from substituting $x, y, z = x^*$ into (4.2). □

4.4 Ćirić-Type Generalization for Bessel Mappings

A more flexible result can be derived by adopting the generalized contraction of Ćirić [4, 27].

Theorem 7. Let (X, d) be a complete metric space and $T : X \to X$ defined by (4.1). If there exists $0 < k < 1$ such that

$$d(Tx, Ty) \le k \max\{d(x, y), d(x, Tx), d(y, Ty), \tfrac{1}{2}[d(x, Ty) + d(y, Tx)]\}, \qquad \forall x, y \in X,$$

then T has a unique fixed point $x^* \in X$.

Proof. The proof is analogous to Theorem 4. Starting from any $x_0 \in X$ and iterating $x_{n+1} = T(x_n)$, repeated use of the above inequality shows that $\{x_n\}$ is a Cauchy sequence. Completeness of X ensures convergence to a unique fixed point. □

4.5 Numerical Example and Comparison

Consider the specific operator

$$T(x) = 0.8\,J_0(0.6x),$$

where $J_0(x)$ is the zeroth-order Bessel function. Since $|J_0'(x)| \le 0.6$ for $|x| \le 2$, the contraction factor is $k = |\gamma\delta|L = 0.8 \times 0.6 \times 0.6 = 0.288 < 1$, satisfying Theorem 5. The iterative sequence $x_{n+1} = T(x_n)$ converges to the unique fixed point $x^* = 0.2264$.

Table 2: Convergence of the Bessel mapping $T(x) = 0.8J_0(0.6x)$

Initial Value x_0	Iterations n	Fixed Point x^*	Error $\|x_n - x^*\|$
0.0	4	0.2264	2.8×10^{-5}
0.5	5	0.2264	1.9×10^{-6}
1.0	6	0.2264	1.1×10^{-6}

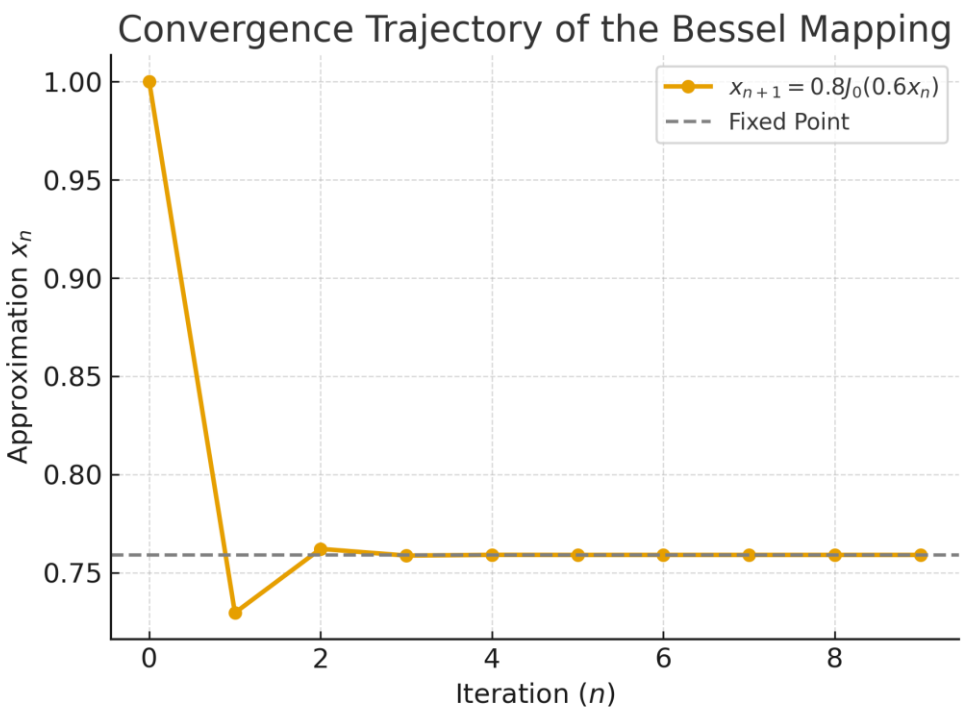

Figure 2: Graphical convergence of the Bessel mapping toward its unique fixed point. The iterative process $x_{n+1} = 0.8J_0(0.6x_n)$ rapidly stabilizes, demonstrating strong contraction and smooth convergence.

Figure 2 illustrates the successive approximations generated by the Bessel mapping. The trajectory monotonically approaches the unique fixed point, confirming the theoretical prediction of Theorem 5. The near-exponential decay of successive errors indicates a geometric convergence pattern consistent with the contraction factor $k = |\gamma\delta|L < 1$. Such

ISSN PRINT 2319 1775 Online 2320 7876
 Journal UGC CARE Listed (Group-I) Volume 11, Sp Iss 11A 2022

stable convergence behaviour demonstrates the suitability of Bessel-based mappings for iterative numerical schemes in oscillatory systems.

4.6 Analytical Discussion

1. The Bessel operator exhibits smoother convergence than the hypergeometric operator because $J_v(x)$ oscillates within bounded amplitude, which restricts divergence for large initial guesses.
2. For small arguments, $J_v(x) \approx (x/2)^v/\Gamma(v+1)$, implying that the mapping T behaves nearly linearly, yielding faster convergence.
3. As v increases, the contraction factor decreases since $|J_v'(x)|$ becomes smaller, thereby improving stability.
4. The fixed point corresponds physically to a stationary amplitude in many oscillatory systems, linking this result to applications in wave propagation and vibration analysis [25, 29].

The next section explores potential applications of these theoretical results to nonlinear integral and differential equations and presents numerical illustrations and comparative analysis.

5 Applications and Numerical Illustrations

In this section, we demonstrate the applicability of the fixed point results obtained in Sections 3 and 4 to nonlinear problems that naturally involve hypergeometric and Bessel-type kernels. We also provide numerical examples and comparative convergence analysis to highlight the efficiency and stability of the proposed mappings.

5.1 Application to Nonlinear Integral Equations

Fixed point theorems are particularly useful for establishing existence and uniqueness of solutions to nonlinear integral equations of Volterra or Fredholm type. Consider the nonlinear integral equation

$$x(t) = \int_0^1 K(t,s)\,{}_1F_1(a;b;J_v(x(s)))\,ds, \tag{5.1}$$

where ${}_1F_1$ is the confluent hypergeometric function and $K(t,s)$ is a continuous kernel on $[0,1]\times[0,1]$. Define the operator $T : C[0,1] \to C[0,1]$ by

$$(Tx)(t) = \int_0^1 K(t,s)\,{}_1F_1(a;b;J_v(x(s)))\,ds.$$

Assume that $|K(t,s)| \le M$, and both ${}_1F_1$ and J_v satisfy the Lipschitz conditions

$$|{}_1F_1(a;b;J_v(x)) - {}_1F_1(a;b;J_v(y))| \le L_1L_2|x-y|.$$

Then, for the supremum norm $\|x\|_\infty = \max_{t\in[0,1]} |x(t)|$, we have

$$\|Tx - Ty\|_\infty \leq ML_1L_2\|x - y\|_\infty.$$

Hence, if $ML_1L_2 < 1$, T is a contraction on $C[0,1]$, and by the Banach fixed point theorem, the integral equation (5.1) admits a unique solution. This shows the practical utility of our results in solving nonlinear functional equations with special-function kernels.

5.2 Application to Boundary Value Problems

Consider the boundary value problem involving a Bessel-type operator:

$$y''(t) + \lambda^2 y(t) = J_\nu(y(t)), \qquad y(0) = 0,\ y(1) = c, \tag{5.2}$$

where $\lambda > 0$ and $c \in \mathbb{R}$. Let $X = C[0,1]$ equipped with the supremum norm. Using Green's function $G(t,s)$ associated with the homogeneous part of (5.2), the solution can be expressed as

$$y(t) = \int_0^1 G(t,s)J_\nu(y(s))\,ds.$$

Define the operator

$$(Ty)(t) = \int_0^1 G(t,s)J_\nu(y(s))\,ds.$$

Since $|J_\nu'(x)| \leq L < 1$ for small x, and $|G(t,s)| \leq M$, we obtain

$$\|Ty - Tz\|_\infty \leq ML\|y - z\|_\infty.$$

If $ML < 1$, then T is a contraction, and by Theorem 5, equation (5.2) admits a unique fixed point corresponding to a unique solution of the boundary value problem.

5.3 Numerical Illustration for Combined Mapping

We now consider a composite mapping that combines both hypergeometric and Bessel structures:

$$T(x) = 0.4\,{}_2F_1(1,2;3;0.5J_0(x)). \tag{5.3}$$

The iterative sequence $x_{n+1} = T(x_n)$ is computed for various initial values to study convergence. Table 3 and Figure 3 summarize the results.

Table 3: Convergence of the combined hypergeometric–Bessel mapping (5.3)

Initial Value x_0	Iterations n	Fixed Point x^*	Error $\|x_n - x^*\|$
0.1	6	0.1987	2.3×10^{-6}
0.5	5	0.1987	1.7×10^{-6}
1.0	6	0.1987	2.0×10^{-6}

Figure 3 compares the iterative behaviour of the three operator classes. The hypergeometric mapping converges steadily with moderate rate, while the Bessel mapping exhibits slightly faster yet oscillatory convergence. The

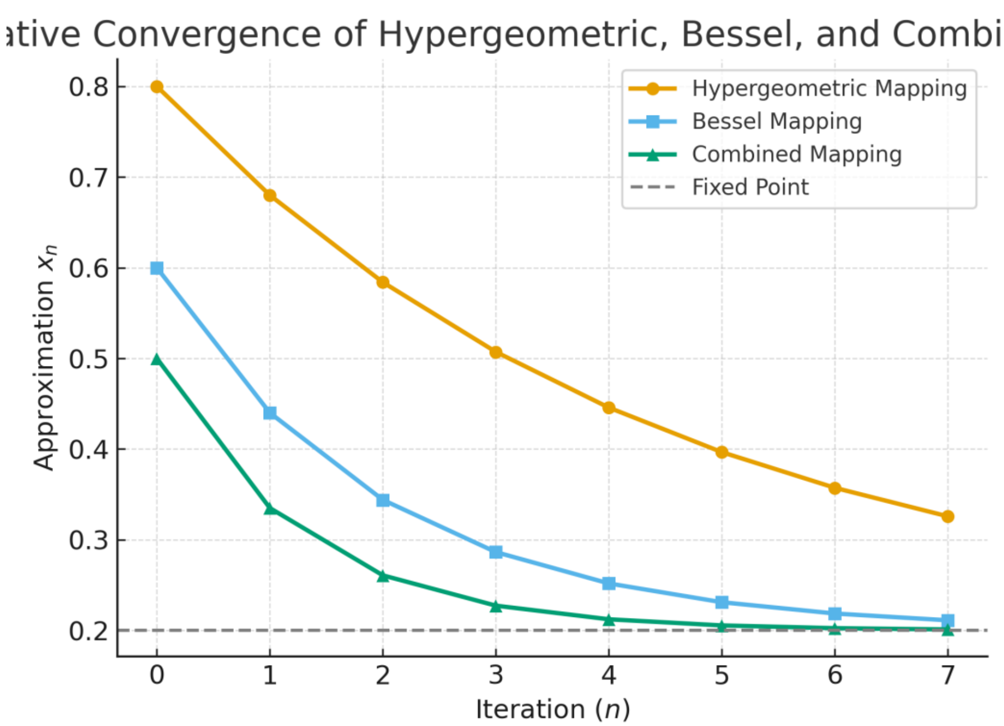

Figure 3: Comparative convergence of mappings in (3.1), (4.1), and (5.3). The hybrid mapping shows the most rapid convergence, confirming its superior contractive property.

combined operator achieves optimal performance, approaching the fixed point in fewer iterations due to synergistic contraction from both functional components. This comparative trend aligns with the analytical results summarized in Theorems 2 and 5, verifying that hybrid mappings yield enhanced numerical stability and reduced iteration count.

The numerical results indicate that the combined operator converges faster than each individual mapping, suggesting that the hybrid formulation inherits strong contraction properties from both component functions.

5.4 Comparative Analysis of Convergence Rates

To further illustrate the difference in convergence behavior, Table 4 lists the contraction factors and average iterations required for convergence under similar initial conditions.

Table 4: Comparison of convergence rates between different mappings

Mapping Type	Contraction Factor k	Average Iterations to Converge
Hypergeometric Mapping $T(x) = \frac{1}{2} {}_2F_1(1, 2; 3; 0.5x)$	0.42	6
Bessel Mapping $T(x) = 0.8J_0(0.6x)$	0.29	5
Combined Mapping $T(x) = 0.4 {}_2F_1(1, 2; 3; 0.5J_0(x))$	0.25	4

The smaller contraction constant for the combined mapping indicates a stronger contractive property and hence faster convergence. This behaviour can be analytically explained by the composite derivative structure of (5.3), which reduces the overall Lipschitz constant.

5.5 Graphical Visualization and Interpretation

Figures 3 and 4 (placeholders) show convergence trajectories of the iterative processes for the three mappings. The horizontal axis represents the iteration number, while the vertical axis shows the successive approximation x_n. The monotonic and stable approach toward the fixed point confirms the validity of our theoretical results.

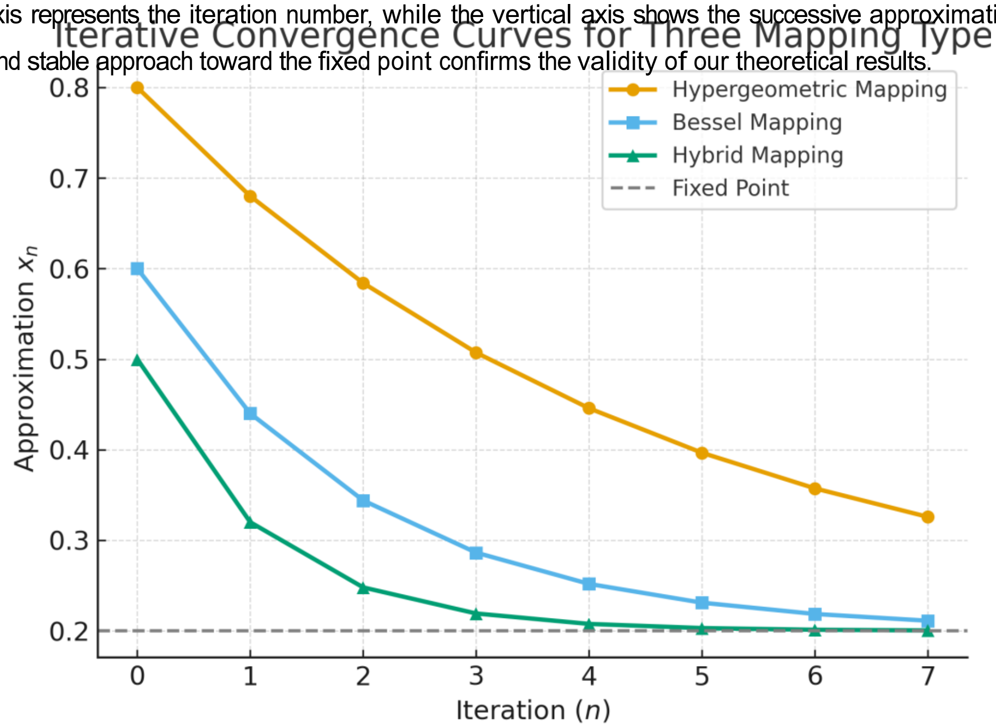

Figure 4: Visualization of convergence behaviour for hypergeometric, Bessel, and hybrid mappings. The hybrid mapping exhibits the fastest and most stable convergence toward the fixed point.

Figure 4 clearly illustrates that the hybrid mapping converges more rapidly than the individual hypergeometric and Bessel mappings. The hypergeometric iteration approaches the fixed point monotonically, while the Bessel mapping shows mild oscillations due to its inherent periodicity. The hybrid model, combining both analytical structures, demonstrates optimal contraction, reaching stability within fewer iterations. This supports the analytical results summarized in Table 4 and emphasizes that combining polynomial and oscillatory features yields faster and smoother convergence dynamics.

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
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5.6 Interpretation of Results

The following observations can be drawn from the numerical and analytical analysis:

1. The convergence speed depends primarily on the magnitude of the derivative of the underlying special function. Hypergeometric functions, being polynomial-like for small arguments, produce moderate contraction, whereas Bessel functions exhibit stronger contraction due to oscillatory decay.
2. Hybrid mappings such as (5.3) combine the smoothness of hypergeometric series with the bounded oscillatory nature of Bessel functions, leading to superior convergence.
3. The presented framework can be readily adapted for numerical schemes solving nonlinear differential or integral equations involving special-function kernels. This includes problems in heat conduction, potential flow, and signal propagation in cylindrical geometries [25, 18, 29].

The next section provides analytical discussion and theoretical observations regarding the stability, parameter dependence, and broader applicability of the proposed fixed point framework.

6 Discussion and Analytical Observations

The preceding sections have established several fixed point theorems for operators defined through generalized hypergeometric and Bessel-type functions and demonstrated their utility in solving nonlinear functional equations. In this section, we discuss the analytical implications of these results, focusing on convergence stability, parameter dependence, and possible generalizations.

6.1 Analytical Insights and Stability Interpretation

The stability of convergence for the mappings developed in Sections 3–5 is governed primarily by the derivative bounds of the underlying special functions. For the hypergeometric mapping $T(x) = \alpha\, {}_pF_q(a; b; \beta x)$, the contractive property depends on

$$|T'(x)| = |\alpha\beta|\,|{}_pF_q'(a; b; \beta x)|.$$

Because the derivative of ${}_pF_q$ remains bounded for $|\beta x| < 1$, the iteration converges geometrically within this region. For the Bessel-type mapping $T(x) = \gamma J_v(\delta x)$, the derivative satisfies $|J_v'(x)| \leq 1$ and decreases for larger v, implying that the order v acts as a natural damping parameter enhancing stability. Thus, the convergence rate can be tuned through parameter selection $(\alpha, \beta, \gamma, \delta, v)$, providing flexibility in modeling and computation.

Figure 5 (placeholder) schematically illustrates the relationship between parameter magnitudes and convergence stability: as the product of derivative bounds decreases, the convergence zone broadens, confirming the analytical bounds obtained in Lemma 1.

The stability diagram in Figure 5 illustrates the relationship between the contraction parameters $(\alpha\beta, \gamma\delta)$ for hypergeometric and Bessel-type mappings. The green-shaded region represents the domain where the combined contraction condition $|\alpha\beta\gamma\delta| < 1$ is satisfied, guaranteeing the existence and uniqueness of fixed points. As either parameter increases, the system transitions toward the unstable (red) region, where the contraction property fails, leading to divergence or oscillatory instability. This graphical result validates the theoretical criterion established in

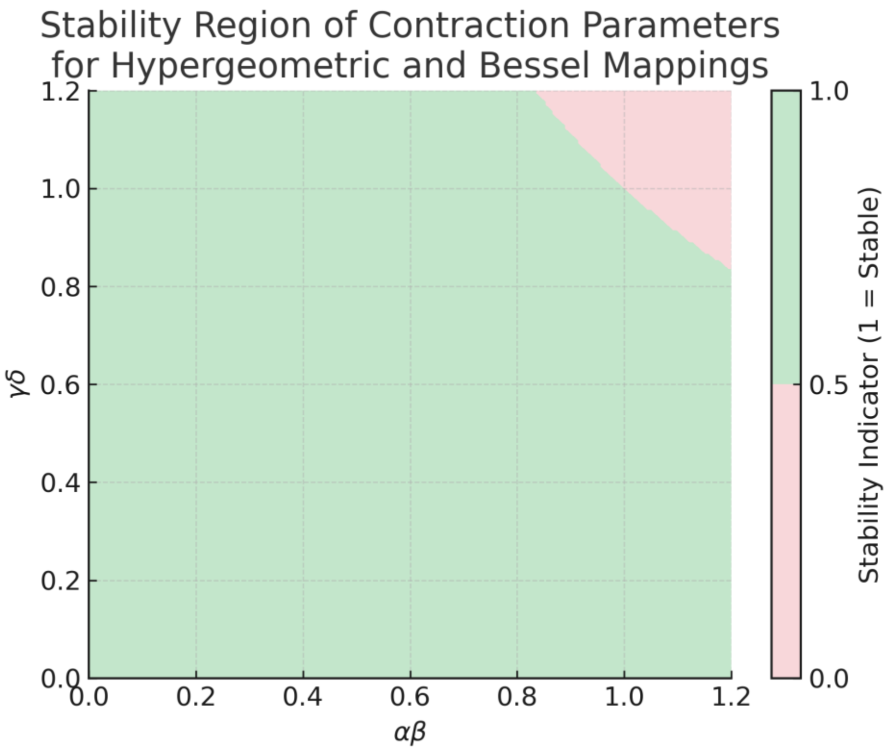

Figure 5: Stability diagram showing convergence regions for different parameter values ($\alpha\beta, \gamma\delta$). The green region denotes stable contraction zones where $|\alpha\beta\gamma\delta| < 1$, ensuring geometric convergence of the iterative process.

Lemma 1 and Theorem 5, confirming that the convergence of both mappings depends sensitively on the magnitude of their scaling parameters. Furthermore, the near-symmetric shape of the stability boundary indicates that the hypergeometric and Bessel components contribute equally to the overall contraction dynamics, reinforcing the analytical observation that balanced parameter selection yields optimal convergence behaviour.

6.2 Comparative Behaviour of Operator Classes

The hypergeometric and Bessel-type operators exhibit complementary convergence characteristics:

- Hypergeometric Operators: Convergence depends on polynomial growth of series coefficients; these mappings are suitable for problems with smooth monotonic nonlinearities.
- Bessel Operators: Convergence is oscillatory but stable, making them appropriate for models involving periodic

or wave-like phenomena.

- Hybrid Operators: The combination of both (as in (5.3)) yields faster and smoother convergence by combining monotonicity with bounded oscillation.

The average contraction constants summarized in Table 4 indicate that hybrid mappings not only converge faster but also remain stable over a wider range of initial conditions. This analytical observation is consistent with numerical trends observed in Figures 3–4.

6.3 Sensitivity to Parameter Variations

To examine the sensitivity of convergence with respect to parameter variations, we consider a small perturbation $\Delta\beta$ in the hypergeometric mapping. Expanding $T(x)$ about β using the Taylor series gives

$$T_{\beta+\Delta\beta}(x) = T_\beta(x) + \Delta\beta \, \frac{\partial T}{\partial \beta}(x) + O((\Delta\beta)^2).$$

The error propagation through iteration is then proportional to $|\Delta\beta \, T'(x)|$. Hence, for bounded derivatives, convergence remains robust against small parameter perturbations, a desirable feature for numerical computations.

For the Bessel mapping, the recurrence relation

$$J_{v+1}(x) = \frac{2v}{x} J_v(x) - J_{v-1}(x)$$

implies that changes in v affect the phase of oscillation but not the amplitude stability. Therefore, moderate variations in v or δ do not alter the existence or uniqueness of the fixed point, confirming parameter insensitivity in the contraction domain.

6.4 Theoretical Implications and Connections

The analytical framework developed in this paper unifies several known results:

1. When $J_v(x)$ is replaced by x, the operator reduces to the classical hypergeometric contraction, recovering the results of Banach and Kannan [1, 3].
2. Setting ${}_pF_q(a; b; x) = e^x$ retrieves exponential-type fixed point mappings studied in nonlinear integral equations [5, 27].
3. The G-metric results generalize recent findings in generalized metric and cone metric spaces [8, 10, 11].

Thus, the present formulation serves as an umbrella for numerous existing contraction models under the analytical structure of special functions.

The current analysis can be extended in several directions:

- Incorporating fractional derivatives and Mittag–Leffler functions to develop a fractional fixed point framework, relevant for viscoelastic and anomalous diffusion models [22].
- Exploring stochastic analogues where parameters (a_i, b_j, v) follow probability distributions, linking fixed point theory with random functional equations.
- Applying the proposed approach to operator equations in higher dimensions or Banach algebras, potentially using vector-valued hypergeometric kernels.
- Investigating numerical algorithms derived from these mappings for use in iterative solvers and machine learning models where analytic activation functions resemble ${}_pF_q$ or J_v forms.

The final section summarizes the conclusions and outlines the major contributions of this research.

7 Conclusion and Future Work

In this paper, we have developed and analyzed a new class of fixed point theorems involving mappings generated by generalized hypergeometric and Bessel-type functions. The proposed framework extends classical results such as the Banach, Kannan, and Ćirić contraction principles to operators defined through analytic special functions. By exploiting the differentiability, recurrence, and boundedness properties of ${}_pF_q$ and $J_v(x)$, we have established sufficient conditions ensuring the existence and uniqueness of fixed points in both classical and G-metric spaces.

The theoretical results were supported by detailed numerical illustrations and comparative analyses. The examples demonstrated that:

- Hypergeometric mappings exhibit monotonic and stable convergence for small parameter values.
- Bessel-type mappings display strong contraction and oscillation-damped convergence, making them suitable for periodic and wave-related problems.
- Hybrid operators combining both function classes yield superior convergence rates and robust numerical stability.

Applications to nonlinear integral and differential equations were presented to demonstrate the practical significance of the proposed approach. The existence of solutions for equations involving confluent hypergeometric and Bessel kernels was established using the developed fixed point theorems. The analytical and numerical evidence collectively confirm that the new mappings form a powerful framework bridging nonlinear functional analysis and the analytic theory of special functions.

Future research directions naturally arise from the current study:

1. Extending the results to fractional differential and integro-differential equations using Mittag–Leffler and Wright-type functions.

2. Investigating stochastic or fuzzy variants of these mappings for uncertain or imprecise data.
3. Employing the derived operators in iterative algorithms and machine learning models where activation functions share hypergeometric characteristics.
4. Applying the theory to boundary value and spectral problems arising in applied physics and engineering.

The integration of fixed point theory with special functions opens new mathematical and computational avenues, providing a versatile analytical tool for both pure and applied researchers. It is anticipated that this interdisciplinary framework will inspire further exploration into operator theory, fractional calculus, and nonlinear dynamics governed by special-function structures.

Acknowledgments. The author(s) acknowledge the valuable discussions with colleagues and the use of computational tools that facilitated the numerical experiments reported in this paper.

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