VARIATIONAL ITERATION METHOD FOR PARTIAL DIFFERENTIAL EQUATIONS

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Abstract
In this paper, we focus on the Variational Iteration Method (VIM) for linear and nonlinear partial differential equations (PDEs). Initially, teach some fundamental concepts of the Variational iteration method and its algorithm for solving PDEs. The Variational iteration approach is also used to solve several linear and nonlinear differential equations of first and second order. The benefit of VIM is that it does not require a tiny parameter in an equation, as would a perturbation approach. The VIM is used to solve a wide range of nonlinear problems in an effective, simple, and precise manner, using approximations that converge quickly to accurate answers. Because the Lagrange multiplier can be precisely recognized, the exact solution to linear problems may be achieved in a single iteration.

KEYWORDS: Variational iteration method, Partial differential equations, Lagrange multiplier.

1. Introduction

Ji-Huan He[1] provided a very straightforward and basic explanation of the variational iteration approach. The variational iteration approach [1-5] is unusually simple, yet the results are surprise accurate (occasionally precise solutions may be achieved). Non-mathematical students can readily understand the method and apply it to a variety of nonlinear situations. Furthermore, many authors worked hard to provide sophisticated theoretical verification of the variational iteration method. For example, Hesameddini and Latifizadeh [6] reconstructed the variational iteration algorithms using the well-known Laplace transform; Odibat [7], Salkuyeh [8], and Tatari & Dehghan [9] demonstrated that the variational iteration algorithm leads to convergent results. The approach has been frequently used to nonlinear models.

The variational iteration method is a dependable analytical tool for solving nonlinear wave equations, according to Wazwaz, who applied the method to a variety of nonlinear systems, such as nonlinear diffusion equations and nonlinear wave equations[10–13]; Goh, et al[14]. The method's flexibility and capacity to solve nonlinear equations accurately and conveniently are its main characteristics. showed that the variational iteration method is a highly promising approach for integro-differential equations; Uremen and Yildirim obtained exact solutions of the Poisson equation [16]; Sadighi and Ganji also obtained exact solutions of nonlinear diffusion equations [17]. All of these studies came to a similar conclusion: the variational iteration method is a reliable treatment for hyperchaotic systems. Numerous variations of the variational iteration method exist [18–21]. Herianu and Marinca's modification (an optimal variational iteration algorithm) [20] is one of the most appealing; it combines the variational iteration method with least-squares technology and produces optimal results after just one iteration. He et al. [6] recently provided a summary of several variational iteration techniques for diverse types of nonlinear equations, such as differential-difference equations, fractal-differential equations, and fractional-differential equations. Even though a lot has been accomplished, Cauchy problems have not yet been solved using the variational iteration technique.

The first-order partial differential equation of the following form is discussed in this work using the following method:

$$u_t(x,t)+a(x,t)u_x(x,t)=\phi(x),\quad x\in R,\quad t\geq 0 \qquad (2.1)$$

$$u_t(x,0)=\phi(x),\ x\in R \qquad (2.2)$$

2. He's Variational method for the first-order partial differential equation

To illustrate the basic concept of the VIM, we consider the following general nonlinear system

$$Lu+Nu=\phi(x) \qquad (2.3)$$

where L is a linear operator part while N is the nonlinear operator part and $\phi(x)$ is a known analytic function.

According to the VIM, a correction function can be constructed as follows

$$u_{n+1}(x)=u_n(x)+\int_0^x \lambda\{Lu_n(\xi)+N\vec{u}_n(\xi)-\phi(\xi)\}d\xi \qquad (2.4)$$

where λ is a general Lagrange multiplier which can be identified optimally via the variational theory, the subscript n denotes the nth approximation and $\bar{u}_n(\xi)$ is considered as a restricted variation

$$i.e.,\quad \delta\bar{u}_n=0$$

The initial guess $u_0(x)$ be freely chosen with a possible unknown constant; it can also be solved from its corresponding linear homogeneous equation

$$Lu=0 \qquad (2.5)$$

Numerous variations of the variational iteration method exist [18–21]. Herianu and Marinca's modification (an optimal variational iteration algorithm) [20] is one of the most appealing; it combines the variational iteration method with least-squares technology and produces optimal results after just one iteration. He et al. [6] recently provided a summary of several variational iteration techniques for diverse types of nonlinear equations, such as differential-difference equations, fractal-differential equations, and fractional-differential equations. Recently, Gouder et al. [22] talked about using the Homotopy perturbation approach to solve the wave equation. Even though a lot has been accomplished, Cauchy problems have not yet been solved using the variational iteration technique.

The first-order partial differential equation of the following form is discussed in this work using the following method:

$$u_{n+1}(x) = u_n(x,t) + \int_0^t \lambda\{\frac{\partial u_n(x,\xi)}{\partial \xi} + a(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} - \phi(x)\}d\xi \qquad (2.6)$$

The stationary conditions are given by

$$1+\lambda=0 \qquad \lambda'|_{\xi=t}=0 \qquad (2.7)$$

So that λ=-1. As a result, we obtain the following iteration formula

$$u_{n+1}(x) = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,\xi)}{\partial \xi} + a(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x} - \phi(x)\right\}d\xi \qquad (2.8)$$

3. Applications of Variational iteration method

Example-1: Consider the transport equation

$$u_t(x,t) + au_x(x,t) = 0, \quad x \in R, \quad t > 0$$
$$u(x,0) = x^2, \qquad x \in R \qquad (2.9)$$

According to this iteration formula (2.8), we have

$$u_{n+1}(x) = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,\xi)}{\partial \xi} + a(x,\xi)\frac{\partial u_n(x,\xi)}{\partial x}\right\}d\xi \qquad (2.10)$$

We can select $u_0(x,t) = x^2$ from the given initial condition .putting this selection into Eq.(2.10), we obtain the following successive approximations

$$u_0(x,t) = x^2$$

$$u_1(x,t) = u_0(x,t) - \int_0^t \{\frac{\partial u_n(x^2)}{\partial \xi} + a\frac{\partial u_n(x^2)}{\partial x}\} d\xi$$

$$u_1(x,t) = x^2 - 2atx$$

$$u_2(x,t) = x^2 - 2atx + a^2t^2 \qquad (2.11)$$

....

$$u_n(x,t) = x^2 - 2atx + a^2t^2$$

Hence we obtain the exact solution : $u(x,t) = x^2 - 2atx + a^2t^2$.

Example-2: Consider the nonlinear Cauchy-problem

$$u_t(x,t) + xu_x(x,t) = 0, \quad x \in R, \quad t > 0$$

$$u(x,0) = x^2, \quad x \in R \qquad (2.12)$$

According to the formula of iteration (2.8), we have

$$u_{n+1}(x) = u_n(x,t) - \int_0^t \left\{\frac{\partial u_n(x,\xi)}{\partial \xi} + x\frac{\partial u_n(x,\xi)}{\partial x}\right\} d\xi \qquad (2.13)$$

As stated before, we can select $u_0(x,t) = x^2$ from the given initial condition, using this selection, we obtain the following successive approximations

$$u_0(x,t) = x^2$$

$$u_1(x,t) = x^2 - 2x^2t = x^2(1-2t)$$

$$u_2(x,t) = x^2 - 2x^2t + 2x^2t^2 = x^2(1 - 2t^2 + \frac{(2t)^2}{2}) \qquad (2.14)$$

$$u_t(x,t) = x^2 - 2tx^2 + 2t^2x^2 - \frac{4}{3}t^3x^2$$

$$u_t(x,t) = x^2(1 - 2t^2 + \frac{(2t)^2}{2!} - \frac{(2t)^2}{3!} + \frac{(2t)^2}{4!} - \frac{(2t)^2}{5!} + ...$$

The VIM admits the use of $u(x,t) = \lim_{n\to\infty} u_n(x,t)$.

which gives the exact solution $u(x,t) = x^2e^{-2t}$:

Example-3: Consider the following non-homogeneous Cauchy problem

$$u_t(x,t) + u_x(x,t) = x, \quad x \in R, t > 0$$

$$u(x,0) = e^x, \qquad x \in R \qquad (2.15)$$

According to the iteration formula (2.8)

$$u_{n+1}(x,t)=u_n(x,t)-\left\{\int_0^t \frac{\partial u_n(x,\xi)}{\partial \xi}+\frac{\partial u_n(x,\xi)}{\partial x}-x\right\}d\xi \qquad (2.16)$$

The zeroth approximation is chosen as $u_0(x,t)=e^x$. Accordingly, we obtain the following successive approximations:

$$u_0(x,t)=e^x$$

$$u_1(x,t)=e^x-e^x t+tx$$

$$u_2(x,t)=e^x-e^x t-\frac{t^2}{2}+\frac{e^x t^2}{2}+tx$$

$$u_x(x,t)=t(x-\frac{t}{2})+e^x(1-t+\frac{t^2}{2!})$$

$$u_2(x,t)=e^x-e^x t-\frac{t^2}{2}+\frac{e^x t^2}{2}-\frac{e^x t^2}{6}+tx \qquad (2.17)$$

$$u_x(x,t)=t(x-\frac{t}{2})+e^x(1-t+\frac{t^2}{2!}-\frac{t^3}{3!})$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

$$u_n(x,t)=t(x-\frac{t}{2})+e^x(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\frac{t^4}{4!}-\frac{t^5}{5!}+\dots\dots)$$

The VIM admits the use of $u=\lim_{n\to\infty} u_n$ and which gives the exact solution

$$u_x(x,t)=t(x-\frac{t}{2})+e^{x-t}$$

Example-4: Consider the inhomogeneous nonlinear system

$$u_t+vu_x+u=1$$

$$v_t+uv_x-v=1 \qquad (2.19)$$

$$u(x,0)=e^x,\quad v(x,0)=e^{-x},\quad x\in R$$

The correction functional for (2.19) real

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda_1\left(\frac{\partial u_n(x,\xi)}{\partial \xi}+\tilde{v}_n(x,t)\frac{\partial u_n(x,\xi)}{\partial x}+\tilde{u}_n(x,\xi)-1\right)d\xi$$

$$v_{n+1}(x,t)=v_n(x,t)+\int_0^t \lambda_2\left(\frac{\partial u_n(x,\xi)}{\partial \xi}+\tilde{u}_n(x,t)\frac{\partial u_n(x,\xi)}{\partial x}+\tilde{v}_n(x,\xi)-1\right)d\xi \qquad (2.20)$$

The zeroth approximations $u_0(x,t)=e^x$ and $v_0(x,t)=e^{-x}$ are selected by using the given initial conditions. The following successive approximations

$$u_0(x,t)=e^x$$

$$v_0(x,t)=e^{-x}$$

$$u_1(x,t)=e^x-te^x$$

$$v_1(x,t)=e^{-x}+te^{-x}$$

$$u_2(x,t)=e^x-te^x+\frac{t^2}{2!}e^x$$

$$v_2(x,t)=e^{-x}+te^{-x}+\frac{t^2}{2!}e^{-x}$$

$$u_3(x,t)=e^x-te^x+\frac{t^2}{2!}e^x-\frac{t^3}{3!}e^x \qquad (2.20)$$

$$v_3(x,t)=e^{-x}+te^{-x}+\frac{t^2}{2!}e^{-x}+\frac{t^3}{3!}e^{-x}$$

....

....

$$u_n(x,t)=e^x\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+........\right)$$

$$v_n(x,t)=e^{-x}\left(1+t+\frac{t^2}{2!}+\frac{t^3}{3!}+........\right)$$

Follow immediately, this gives the exact solutions

$$u(x,t)=e^xe^{-t}=e^{x-t}$$

$$v(x,t)=e^{-x}e^{t}=e^{-x+t}$$

4. Conclusion

This study uses the most prevalent mathematical physics issues to illustrate how VIM may be applied to solve a class of IVPs. Solutions with readily calculable terms in the form of convergent series are produced by this approach. Numerical examples demonstrate how using the VIM can, after a few iterations, lead to accurate answers. In comparison to Adomian's technique, it can be argued that VIM is a very strong and user-friendly instrument for solving nonlinear IVPs. The findings clearly show that the variation iteration approach gives us an effective instrument for finding precise solutions to differential equations. Unlike the perturbation approach, this method does not need tiny parameters in the equation. The main part of the methods is the construction of correction functional which can easily be constructed by the Lagrange multiplier and the multiplier can optimally be identified by variational theory. The application of restricted variations in correction functional makes it much easier to determine.

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