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SPECTRAL iGRAPH iTHEORY iON iPRE iFRACTALS

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Introduction

Fractal sets are “too rough" to define the classical Laplace operator as is the case for open subsets of R^n. We can still however define the Laplace operator on fractals through various approaches, such as a probabilistic one studied by and an analytic one, via the use of Dirichlet forms, originating from the work of Kigami .In this thesis we will focus on Kigami’s analytic approach. We will use the slightly different convention of, that is the same in spirit to that of Kigami, but is perhaps more suited for considering things from the graph theoretic point of view. On the graph approximation G_n, we set $V_n = \cup_i F_i V_{n-1}$ and $V_* = \cup_i V_i$. Now, for functions $u, v : V_n \longrightarrow R$ we can define the energy form

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$$\mathcal{E}_n(u,v) = \sum_{y \sim_n x} c_n(x,y)(u(x)-u(y))(v(x)-v(y))$$

where the notation $y \sim_n x$ denotes that the vertices x and y are adjacent in the graph G_n and $c_n(x, y)$ is the conductance of the edge connecting the vertices x and y. We want to create an energy form on the fractal K and whether that can be done is a difficult problem depending on K and the choices of $c_n(x, y)$. Some requirements need to be satisfied, but for our purposes here we will not focus on this renormalizaation problem. However, we will instead focus here in the cases where this is possible and all the conductances satisfy $c_n(x, y) = r^{-n}$ where r is the so-called renormalization constant. We will also restrict our attention in the cases where $0 < r < 1$ which we refer as a regular harmonic structure. We denote $E_n(u) = E_n(u,u)$. If we have a function $u : V_n \rightarrow R$ there exists a unique way to extend it to V_{n+1} such that its energy is minimized. This is called the harmonic extension, and in that case $E_{n+1}(u) = E(u)$. So, given a function with initial values on V_0 there exists a unique way to extend it harmonically at every level. Such a function will be called harmonic function. Then the energy of a function $u : K \rightarrow R$ is given by

$$\mathcal{E}(u) = \lim_{m \rightarrow \infty} \mathcal{E}_m(u)$$

We will study functions of finite energy, i.e. the vector space

$$dom\mathcal{E} = \{u : K \rightarrow \mathbb{R} : \mathcal{E}(u) < \infty\}$$

It can be seen that functions of finite energy are continuous and thus since K is compact, uniformly continuous. The space of functions of finite energy modulo constants is a Hilbert space with the energy inner product.

Renormalization and harmonic functions are connected with electric net works and random walks on graphs. Having the standard energy form E on a graph G, we can define the effective resistance between two vertices as

$R(u,v)=(min\{E(h) : h(u) = 0 \text{ and } h(v) = 1\})^{-1}$.

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The function satisfying that minimum is in fact going to be the harmonic func tion with those two boundary values. The effective resistance, also called as resistance metric is a metric on the graph. The probabilistic interpretation of harmonic functions is interpreting the values on the vertices of the graph as probabilities of a random walk on the graph. Take a subset $S \subset G$ of vertices, and denote it as the boundary of the graph. Then if we solve the Dirichlet problem on the graph

$\Delta h(v) = 0$ for all $v \in G \setminus S$ and $h_{|S} = g$

we will have that the harmonic function h can be written as

$$h(u) = \sum_{v \in S} g(v) Prob(u \to v)$$

where $Prob(u \to v)$ denotes the probability that a random walk on the graph G starting at u first hits v before any other vertices of S. We refer the reader for more details to [9, 15].

If $w = (w_1,...,w_n)$ is a finite word, where $w_i \in \{1,...,m\}$ we define the map $F_w = F_{w1} \circ \cdots \circ F_{wn}$. We call $F_w K$ a cell of level $n = |w|$, where $|w|$ is the length of the word. We can now construct a measure on our self similar set K. The standard measure is a special case of a self-similar measure created in the following way. Assign probability weights μ_i with $\sum^{m}_{i=0} \mu_i = 1$ with each $\mu_i > 0$ and set $\mu(F_w K) = \prod^{|w|}_{i=0} \mu_{wi}$. For the standard measure we just assign all $\mu_i = 1/m$. On K for the standard invariant measure μ we have $\mu(F_w F_i K) = \frac{1}{m}\mu(F_w K)$, $i = 0,1,2,...,m$ for any word w. The self-similar identity $\mu(A) = \sum i\ \mu_i \mu(F^{-1} A)$ holds for set $A \subset K$. In fact, the standard mea sure is none other than a renormalized version of the Hausdorff measure. Using the measure μ we can study integrals so as to do analysis on the fractal.

Our functions are uniformly continous because the set K is compact, so we simply define integration as

$$\int_K f d\mu = \lim_{m \to \infty} \sum_{|w|=m} f(x_w) \mu(F_w K)$$

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Having the energy form and the integrals at our disposal, we can now define the Laplace operator on the fractal itself.

Definition 3.2.1. *Let* $u \in dom\mathrm{E}$ *and* f *be a continuous function. Then,* $u \in dom\Delta_\mu$ *and* $\Delta_\mu u = f$ *if*

$$\mathcal{E}(u,v) = -\int_K fv\,d\mu \text{ for all } v \in dom\mathcal{E}_0$$

where $dom\varepsilon_0$ *denotes the functions of finite energy that vanish on the bound ary.*

Introducing the notion of normal derivatives we can modify the above no tion to define the Neumann Laplacian as well. This is refered to as the weak definition of the Laplacian, we may also construct a pointwise definition for the standard self-similar measure by using the graph Laplacians. Specifically, we define the combinatorial graph Laplacian as

$$\Delta_m u(x) = \sum_{y \sim_m x} (u(y) - u(x)) \text{ for } x \in V_m \setminus V_0.$$

The harmonic functions then at every level satisfy $\Delta_m h = 0$. Notice that our definition of harmonic functions is in fact slightly different than the pure graph theoretic version often found in the literature. In spectral graph theory the har monic functions are those that $\Delta h = 0$, where Δ is the graph Laplacian defined above as $\Delta = D - A$, and on finite graphs are only piecewise constant on each connected component of the graph. This difference comes from the fact that the graph Laplacian on the fractal graphs was not defined on the boundary, so as to allow us flexibility when it comes to boundary conditions such as Dirich let or Neumann. Here in the fractal setting, by harmonic functions we refer to the ones solving the Dirichlet problem

$\Delta_m h = 0$ for all $m \geq 1$ and $h_{|}{}^{V}{}_{0} = g$

and thus do not necessarily satisfy the Laplace equation, in the graph theoretic sense, on the boundary. There exists an algorithmic approach to calculating the values of a harmonic function on V_m. This is a local extension algorithm, meaning that knowing the boundary values on any cell

of level m we can extend it to a cell of level $m + 1$ and thus inductively everywhere. Starting from the boundary values at V_0 we can solve the linear system of equations at V_1 giving us the harmonic extension on the first level. Then at every next level, we are in the same situation as before with m cells and new boundary values. Encoding this information into m matrices A_i and thinking of the values of h on V_0 as a vector we have that $h_{|Fi}V_0 = A_i h_{|V0}$ which then inductively gives us that the values on any given cell are obtained by

$$h_{|Fw}V_0 = A_w h_{|V0} \text{ where } A_w = A_{wm} \cdots A_{w2}A_{w1}\,.$$

These are called harmonic extension matrices, and for example in the Sier- pinski gasket they are

$$A_0 = \begin{pmatrix} 1 & 0 & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \end{pmatrix}, A_1 = \begin{pmatrix} \frac{2}{5} & \frac{2}{5} & \frac{1}{5} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \end{pmatrix}, A_2 = \begin{pmatrix} \frac{2}{5} & \frac{1}{5} & \frac{2}{5} \\ \frac{1}{5} & \frac{2}{5} & \frac{2}{5} \\ 0 & 0 & 1 \end{pmatrix}$$

giving us the so-called "$\frac{1}{5} - \frac{2}{5}$" harmonic extension rule, meaning that on m+1 level the value of a harmonic function is $\frac{2}{5}$ times the value of the sum of its closest vertices plus $\frac{1}{5}$ the value of the opposite vertex. Harmonic functions satisfy the maximum principle, meaning that the maximum and minimum val ues are on the boundary V_0. If the matrices A_i are invertible then we have a non-degenerate harmonic structure which implies that non-constant harmonic functions cannot be locally constant on any cell. This is not always the case, for example the Vicsek set in Figure 1.2 has a degenerate harmonic structure.

Now, for $x \in V_m$ let $\psi_x^{(m)} : K \to \mathbb{R}$ defined as $\psi_x^{(m)}(x) = 1$, $\psi_x^{(m)}(y) = 0$ for $y \in V_m \setminus \{x\}$ and then extended harmonically to K. The pointwise definition for the Laplacian then becomes

$$\Delta_\mu u(x) = \lim_{m\to\infty} \left(\int_K \psi_x^{(m)} \mathrm{d}\mu \right)^{-1} \Delta_m u(x).$$

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with uniform convergence on $V_* \backslash V_0$. When we will be using the standard self-similar measure μ we will omit it from the notation and simply write Δ. Then, the harmonic functions that we defined above are exactly the ones such that $\Delta h = 0$. Now, it may not be at this point clear that the space $dom\Delta_\mu$ is in fact a rich space to study. However, indeed $dom\Delta_\mu$ contains a lot of functions. We will omit the details of the construction of Green's function, but through it we can use the following theorem which shows us the richness of $dom\Delta_\mu$.

Theorem 3.2.2. *The Dirichlet problem*

$$-\Delta_\mu u = f, u_{|V0} = 0$$

has a unique solution in $dom\Delta_\mu$ *for any continuous* f, *given by*

$$u(x) = \int_K G(x,y) f(y) \mathrm{d}\mu(\mathrm{y})$$

where G(x,y) is the Green's function. If we don't have Dirichlet boundary conditions then the solution is given by

$$u(x) = \int_K G(x,y) f(y) \mathrm{d}\mu(\mathrm{y}) + h(x)$$

where h(x) is a harmonic function with the same boundary values as u.

One big disadvantage of $dom\Delta_\mu$ is that it's not closed under multiplication. Specifically, for $u \in dom\Delta_\mu$ it is proven in [6] that $u^2 \notin dom\Delta_\mu$. This how ever is specifically for the measure μ. We can define different measures that will give rise to a different Laplacian that will not necessarily suffer from this drawback. Specifically, Kusuoka in [31, 32] defined the measure ν, now re ferred to as the Kusuoka measure. We take a look first at the energy measures of a function $u \in dom\mathrm{E}$. Let the energy measure ν_u be

$$\nu_u(F_w K) = r^{-|w|}\mathrm{E}(u \circ F_w).$$

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Then, let the harmonic functions $h_i(q_j) = \delta_{i\,j}$ for $q_j \in V_0$ and $i = 1,\ldots,|V_0|$. If the harmonic extension matrices A_i are invertible then the energy measures ν_{hi} have full support. We define the Kusuoka measure as

$$\nu = \nu_{h_1} + \cdots + \nu_{h_{|V_0|}}.$$

It can be shown that every energy measure is absolutely continuous with re spect to the Kusuoka measure. Moreover, on the Sierpinski gasket the Kusuoka ´ measure ν is singular with respect to the self-similar measure μ. The Kusuoka measure gives rise to the energy Laplacian Δ_ν , defined in the weak sense exactly as before, now integrating against the Kusuoka measure. However, this energy Laplacian lacks scaling self-similarity and becomes a significantly harder object to study.

Now, for example if $|V_0| = 3$ such as in the case of SG_k, we can also define the Kusuoka measure in terms of an orthonormal basis of harmonic functions $\{h_1,h_2\}$−modulo constants. This gives the same version of the measure as above up to renormalization with a constant and in fact is independent of the choice of orthonormal basis. Then we have that if $u \in dom\Delta_\nu$ then

$$\Delta u^2 = 2u\Delta_\nu u + 2\frac{d\nu_u}{d\nu}$$

is the Radon-Nikodym derivative. This is among one of the reasons that the energy Laplacian is of interest to study and therefore despite some of its disadvantages such as the lack of self-similarity, it behaves better in some regards. An important formula connecting energy measures, energy forms and integration of functions u, $v \in dom\varepsilon$ is the carré du champs formula

$$\int_K f\,d\nu_{u,v} = \frac{1}{2}\mathcal{E}(fu,v) + \frac{1}{2}\mathcal{E}(u,fv) - \frac{1}{2}\mathcal{E}(f,uv)$$

There has also been a study of the Kusuoka measure from the ergodic point of view.

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3.3 Regarding the spectrum

It is interesting to obtain explicit knowledge of the spectrum of the Laplace op erator with respect to various boundary conditions, such as the Dirichlet and Neumann. Depending on the measure used this may not always be possible. For example, the Kusuoka measure gives rise to a Laplace operator that, at the time of this writing, its spectrum is unknown. However, for the self-similar measure, we are able to use its discrete graph approximations to study the spectrum on the actual fractal. This technique is called spectral decimation, first studied in [5, 37], and then in considerable more detail by Fukishima and Shima for the *d*-dimensional Sierpinski gaskets.

It was later expanded significantly to a wide class of p.c.f fractals . For self similar sets of a specific type, see more, we have that the spectrum is obtained through a technique called spectral decimation which can roughly be described as follows. The spectrum of the Laplace operator on the fractal is given by a renormalized limit of a rational function pre-images of eigenvalues on the discrete sequence of graphs approximating it. Moreover, the spectrum on the graph G_{n+1} can be decomposed into "initial eigenvalues" which may ap pear at every level, and continued eigenvalues which are pre-images of eigen values at G_n under the rational function, which is often in fact a polynomial. The set of initial eigenvalues is finite, and the rational function is fixed. This means that every eigenvalue on the graphs is either one of the initial eigen values, or a pre-image of an initial eigenvalue. This approach allows us also to construct the eigenfunctions recursively, but some care needs to be taken because not every pre-image of an eigenvalue is allowed to be taken, we have the so-called forbidden eigenvalues. For more details we refer. So for the fractal itself, as in, we say that a fractal Laplacian admits spectral decimation if its eigenvalues are of the form

$$\lambda = c^m \lim_{n\to\infty} c^n R^{-n}(w)$$

where $w \in W$ is the finite set and the branches of the pre-images are taken in such a way such that the limit exists. We will describe very briefly the process here for simplicity only on the Sierpinski gasket for the combinatorial graph ´ Laplacian with Neumann boundary conditions.

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The Neumann boundary condition corresponds to imagining that the graph is embedded in a larger graph by reflecting in each boundary vertex and using the eigenvalue equation on the even extension. The spectral decimation poly nomial is given by $R(z) = z(5-z)$. The exceptional set of forbidden eigenval ues is $E = \{2,5,6\}$. In multiset notation, on G_0 the spectrum of $\Delta_0 = \{0,6,6\}$. On G_1 it is $\{0,3,3,6,6,6\}$. In particular, at level n every eigenvalue of Δ_n is ei ther 0 or 6 or obtained as a pre-image under R of these, under the condition that we do not encounter a forbidden eigenvalue. We have that $R^{-1}(\{0\}) = \{0,5\}$ and $R^{-1}(\{6\}) = \{2,3\}$ and these are the only cases when we encounter for bidden eigenvalues. Then the spectrum at level n is given by

$$\sigma(\Delta_n) = \{0,6\} \cup \bigcup_{i=0}^{n-1} R^{-i}(\{3\}) \cup \bigcup_{i=0}^{n-2} R^{-i}(\{5\})$$

The eigenvalue 0 is always a simple eigenvalue and the eigenvalues 5 and 6 appear at every level and with high multiplicities. When we take the pre images we have two branches

$$\lambda_n = \frac{5 + \varepsilon_n \sqrt{25 - 4\lambda_{n-1}}}{2}$$

where $\varepsilon_n \in \{-1,1\}$. For the Laplace operator $-\Delta$ on the fractal itself, we can define

$$\lambda = \frac{3}{2} \lim_{n \to \infty} 5^n \lambda_n$$

where the limit in the sequence $\{\lambda_n\}_{n \geq n0}$ is taken for all but a finite number of $\varepsilon_n = -1$, and n_0 is the generation of birth. This procedure gives us the spectrum.

In the year 1966, Kac presented a paper with the title of the now famous question "Can one hear the shape of a drum?". This question is interpreted to mean whether knowledge of the eigenvalues $\{\lambda_n\}$ of the Dirichlet Laplace operator Δ on some bounded domain $U \subset \mathrm{R}^d$ is enough to determine the ge ometry and shape of the domain. The answer to this question is negative, i.e. there exist isospectral domains even in R^2, however this question has moti vated a large amount of

research focusing on heat kernels and the eigenvalue counting function. We define the eigenvalue counting function as

$$N(x) = \#\{n \in \mathrm{N} : \lambda_n \leq x\}.$$

A famous result by Weyl is that for sufficiently regular bounded open domains we have

$$\lim_{x\to\infty} \frac{N(x)}{x^{\frac{d}{2}}} = \frac{\omega_d}{(2\pi)^d} V$$

where V is the volume of the domain and ω_d the volume of the unit ball in R^d. This essentially means that by hearing the drum, while it doesn't give us knowledge of its exact shape, we still obtain some information about its geometry such as its d-dimensional volume.

In analysis on fractals, there is a sharp contrast with that of R^d. Specifically, it was shown in [30] that the situation for the analogue of Weyl's result is dif ferent. For example, for the Dirichlet or Neumann Laplacian on the Sierpinski ´ gasket, it was shown in [30] that there exists a function G such that $0 < \inf G < \sup G < \infty$ and $\frac{\log 5}{\log 2}$-periodic discontinuous $N(x) = x^{\frac{d_s}{2}} G(\log x/2) + O(1)$

giving us that the limit $\lim_{x\to\infty} N(x)x^{-\frac{d_s}{2}}$ does not converge. It is also inter esting to note that the term d_s is different than what we may have expected considering the R^n case where the equivalent term is that of the dimension n. Now, d_s referred to as the spectral dimension, is actually different than the Hausdorff dimension. These quantities are connected via the Einstein relation $2d_s = d_h d_w$ where d_h is the Hausdorff dimension, d_s the spectral dimension and d_w the walk dimension. For more details we refer the reader to [17].

There exists another notable difference between R^n and analysis on fractals. In analysis on fractals we have the existence of joint Dirichlet-Neumann eigen values and specifically, localized eigenfunctions. In fact, the initial eigen values are usually joint Dirichlet-Neumann ones and appear with very high 23 multiplicities. The reason for the high multiplicities is that if the support of an eigenfunction is in a very small cell, we can essentially move it around to create many of

them. That is why they are also referred to as localized eigenfunctions. Their existence is in sharp contrast to analysis on R^n because eigenfunctions are functions that are analytic and therefore cannot be zero on open sets.

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Spectral decimation as in [2, 36] for fully symmetric fractals is valid in gen eral for the probabilstic graph Laplacian and not the combinatorial one. The Laplace operator is usually studied under Neumann or Dirichlet boundary con ditions and these conditions affect the spectrum. If the graph approximations consist of *k*-regular graphs, or perhaps non-regular graphs on the boundary that become regular under the Neumann conditions, then we may also use for our purposes the combinatorial graph Laplacian since it is essentially the same up to a *k* renormalization with the probabilistic graph Laplacian so again we have spectral decimation. We can then define $L = \lim_{n\to\infty} c^n L_n$ where the $c > 1$ is the so-called time-scaling factor and $L_n u(x) = \frac{1}{deg(x)} \sum_{y\sim x}(u(x)-u(y)$

the probabilistic graph Laplacian on the graph G_n. Let $L = -\Delta$. Its spectrum is discrete with eigenvalues of the form

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots < \infty$$

with ∞ the only accumulating point and 0 being an eigenvalue only in the Neumann case corresponding to the constant functions. We are interested in assigning meaning to the product $\prod_{n=1}^{\infty} \lambda_n$ which would be the determinant of the operator. Of course this product is actually infinite, but we would still like to interpret it as a real value.

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In general, some meaning may be given to infinite divergent sums through the process of regularization. For example, it is clear that the sum

However it is useful sometimes, for example in theoretical physics, to attempt to assign real values to such a sum. We know that for $|z| < 1$ it holds that $\sum^{\infty}_{n=0} z^n = \frac{1}{1-z}$ and through the uniqueness of meromorphic continuation we can abuse notation and evaluate the meromorphic continuation of the func tion $f : B(0,1) \rightarrow C$, $f(z) = \sum^{\infty}_{n=0} z^n$ as $\tilde{f}(z) = \frac{1}{1-z}$ at $z = 2$ to obtain that $\sum^{\infty}_{n=0} 2^n = -1$. Of course, this is only a formal expression. Something similar to the above can also be done for $\prod^{\infty}_{n=1} \lambda_n$ using the so-called spectral zeta function. The spectral zeta function is defined as $\zeta_L(s) = \sum^{\infty}_{n=1} \frac{1}{\lambda^s_n}$ for $Re(s)$ sufficiently large so that the sum converges and where the 0 eigenvalue is omit ted in the Neumann case. The spectral zeta function may be meromorphically extended to the entire complex plane and its poles are referred to as complex dimensions. Now, under the assumption that the spectral zeta function has no poles on the imaginary axis, we can still assign it a real value through the following formal manipulations.

We can also define the so-called polynomial zeta functions. Let $R(z) = a_d x^d + \cdots + cx$ be a polynomial that has real coefficients with $d \geq 2$ which satisfies $R(0) = 0$ and $R'(0) = c > 1$. The spectral decimation polynomial satisfies these assumptions. We call Φ the entire function that satisfies the functional equation

$$\Phi(\lambda z) = R(\Phi(z)) \text{ with } \Phi(0) = 0, \Phi'(0) = 1.$$

We can now define the so-called polynomial zeta functions as

$$\zeta_{\Phi,w}(s) = \sum$$

$$\Phi(-\mu)=w$$

$$\mu>0$$

which can also equivalently be stated as

μ^{-s}

$n\to\infty \sum_{z\in R^{-n}(w)}(\lambda^n z)^{-s}.$

$\zeta_{\Phi,w}(s) = \lim$

These zeta functions have been defined and studied in [14], [44] and are crucial in the meromorphic extension of the spectral zeta functions of the Laplace operator L. They can be meromorphically extended to the entire complex plane and none of their poles lie on the imaginary axis. Specifically, all their poles are simple and lie on the imaginary line $\mathrm{Re}(s) = \frac{\log d}{\log \lambda}$. We refer the

reader to [14, 44]. There is also another type of zeta functions studied on fractals, those on fractal strings, see [33].

It often is that we have for some constant α that $L = \alpha \lim_n L_n$. For exam ple in the Sierpinski gasket, in the literature we usually have $L = 6\lim_n L_n$. However, it can also be that α varies in the literature. This is mostly based

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on what convention the authors prefer to use. Then, for example in the unit interval, it just is that

$L = -\Delta = -\frac{d}{dx^2} = \lim_n \Delta_n = 2\lim_n L_n.$

These differences in normalization are at first trivial, making absolutely no difference in the development of the general theory. However, they become more important once we start considering spectral zeta functions. This means that now the eigenvalues are truly of the form

$\lambda = \alpha c^m \lim$

$_{n\to\infty}\, c^n R^{-n}(w)$

and thus the spectral zeta functions may differ from each other like $\zeta_1(s) = \alpha^{-s}\zeta_2(s)$

This distinction becomes important when we attempt to investigate connec tions between discrete and regularized determinants. For our calculations, we will be using $\alpha = 1$.

Conclusion

In paper I, we provide a formula to determine the number of spanning trees on the graph approximations of a post critically finite self-similar fractal admit ting spectral decimation. Specifically, it is shown that the number of spanning trees $\tau(G_n)$ equals

This formula is essentially a calculation of the product of the non-zero eigen values of the probabilistic graph Laplacian. Then, the spanning tree evalu ation is based on Kirchhoff's Matrix-Tree theorem in its probabilistic graph Laplacian version. Moreover, we provide a proof showing why the asymp totic complexity constant c exists for the fractal graphs based only on their self-similarity without using the machinery of [35] as well as also provide the following lower and upper bounds for it.

We conclude the paper with a plethora of examples of specific fractal graphs where we calculate the number of the spanning trees of their graph approxi mations.

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