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The Controllability and Observability of Sylvester Matrix Dynamical Systems on Time Scales

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Abstract In this paper, a solution for the first-order matrix dynamical system is derived, and a set of necessary and sufficient conditions for complete controllability and complete observability of the Sylvester matrix dynamical system is presented.

1. INTRODUCTION

The significance of Sylvester matrix and Lyapunov matrix differential equations and their presence in numerous domains of applied mathematics, including control systems, dynamic programming, optimal filters, quantum mechanics, and systems engineering, is widely recognized. Hence, the two primary objectives of this paper are: (1) the formulation of theory and methodologies for solving dynamical systems on time scales, and (2) the exploration of techniques related to controllability and observability. This paper primarily centers on first-order Sylvester matrix dynamical systems of the form.

$$X^{\Delta}(t) = A(t)X(t)+X(t)B(t)+\mu(t)A(t)X(t)B(t) +C(t)U(t)D*(t),$$

$$X(t_0) = X$$

$$Y(t) = K(t)X(t)L*(t)$$

The given matrix equation involves matrices and functions in various dimensions. Specifically, it consists of a matrix equation of the form, where X(t) represents an n×n matrix, U(t) signifies an m×n input piecewise rd-continuous matrix referred to as the control, and Y(t) corresponds to an r×n output rd-continuous matrix. In this context, the matrices A(t), B(t), C(t), and K(t) hold dimensions of n×n, n×n, n×m, and r×n respectively. Furthermore, D(t) and L(t) are n×n rd-continuous matrices. The symbol XΔ denotes the generalized delta derivative of X, as per Definition 1.10 in reference [2]. The parameter t emanates from a time scale T, characterized as a non-empty closed subset of R, with μ representing the graininess function.

When the condition is met that B = A∗ (where * denotes the transpose of a matrix), the equation (1.1) is labeled as a matrix Lyapunov dynamical system on time scales. Prior works by various authors [4, 11] have successfully derived criteria for complete controllability and complete observability for analogous systems of the type (1.1) and (1.2), with the stipulation that B(t) = 0 and both D(t) and L(t) are identity matrices. Notably, X(t) is then regarded as a vector.

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In instances where the time scale T is set as R, resulting in μ equating to 0, the equation (1.1) transforms into the form of a Sylvester matrix dynamical system.

$$X^{\Delta}(t) = A(t)X(t)+X(t)B(t)+C(t)U(t)D*(t)$$

If the time scale T = Z, then μ = 1, the system (1.1) become Sylvester matrix delta-difference system of the form (1.4)

ΔX(t) = A(t)X(t)+X(t)B(t)+A(t)X(t)B(t)+C(t)U(t)D∗(t), where ΔX(t) = X(t+1)−X(t). In the above system, if we put A(t) = A1(t)−In and B(t) = B1(t)−In then system (1.4) becomes Sylvester matrix difference system of the form (1.5) X(t +1) = A1(t)X(t)B1(t)+C(t)U(t)D∗(t).

Hence, the examination of the controllability behavior within the Sylvester matrix system (1.1) serves to consolidate the study encompassing (1.3), (1.4), and (1.5), extending this unified investigation to matrix dynamic systems operating on time scales. Fausett [5] conducted a comprehensive study on the analytical, numerical solutions, and control-related aspects of the Sylvester matrix differential system (1.3). The work by Murty [10] delved into the topics of existence, uniqueness, controllability, and observability within the context of the matrix delta-difference system (1.4).

The inception of the calculus of time scales, pioneered by Stefan Hilger [6], was driven by the aim of constructing a theoretical framework capable of harmonizing discrete and continuous analysis. The study of dynamic equations on time scales, a burgeoning field within mathematics, has garnered substantial attention recently. This area of inquiry has illuminated the discrepancies existing between continuous differential equations and discrete difference equations, thus obviating the need to prove a result twice—once for differential equations and once for difference equations.

Bohner and Peterson's seminal introductory work [2, 3] encapsulates the central objective: to establish results for dynamic equations wherein the domain of the unknown function adheres to the concept of a "time scale." The efficacy of this approach becomes evident when examining specific instances. For instance, when the time scale T equates to the set of real numbers (T = R), the general outcome aligns with the solution of an ordinary differential equation. Similarly, for T equivalent to the set of integers (T = Z), the resultant findings mirror those of a difference equation.

The organization of this paper is structured as follows: [describe the organization of the paper, detailing its sections, content, and order of presentation.

In section 2, a thorough exploration of fundamental properties associated with time scales and Kronecker product of matrices is undertaken, accompanied by the development of preliminary outcomes that involve transforming the provided problem into a Kronecker product scenario. The resolution to the corresponding initial value problem is subsequently achieved, and this resolution is expressed in terms of two transition matrices pertaining to

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the systems $X^{\Delta}(t) = A(t)X(t)$ and $X^{\Delta}(t) = B^{*}(t)X(t)$, accomplished through the utilization of the standard technique of variation of parameters [8].

Section 3 delves into the examination of necessary and sufficient conditions, all while adhering to specific smoothness conditions. The focal point of this section revolves around the comprehensive analysis of complete controllability and complete observability.

2. Preliminaries

Since 1988, substantial progress has been achieved in unifying the theories of differential equations and difference equations through the establishment of corresponding results within the framework of time scales. For a comprehensive understanding, further details can be found in the references [2] and [3].

Definition 2.1 designates a nonempty closed subset of R as a time scale, denoted as T. An interval, in this context, refers to the intersection of the given interval with a time scale. To define the forward jump operator σ for t < supT and the backward jump operator ρ for t > infT, the following expressions are employed:

$$\sigma(t) = \inf\{s \in T, s > t\} \in T$$

$$\rho(t) = \sup\{s \in T, s < t\} \in T$$

If t and r belong to the time scale T, and σ(t) equals t, t is classified as right dense; otherwise, it is termed right scattered. Similarly, when ρ(r) equals r, r is denoted as left dense; otherwise, it is referred to as left scattered. The graininess function $\mu(t) : T[0,\infty)$ is formulated as $\mu(t) = \sigma(t) - t$.

Definition 2.2, a function $x : T \to R$ is characterized as right dense continuous (rd-continuous) if it exhibits continuity at every right dense point $t \in R$, and if its left-hand limit exists at each left dense point $t \in R$.

Definition 2.3 introduces the concept of rd-continuity for a mapping $f : T \to X$, where X is a Banach space. This mapping is classified as rd-continuous if two conditions are met: (i) it is continuous at each right dense point $t \in T$, and (ii) at every left dense point, the left-hand limit f(t−) exists.

Furthermore, Definition 2.4 defines $F : T^k \to R$ as an antiderivative of $f : T^k \to R$ if the relationship $F^{\Delta}(t) = f(t)$ holds true for all $t \in T^k$. This definition paves the way for the subsequent formulation of the integral by

$$\int_a^t f(s)\Delta s = F(t) - F(a).$$

3. Main Results

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In this section, the necessary and sufficient conditions for achieving complete controllability and complete observability of the matrix dynamical systems on time scales (2.1) and (2.2) are demonstrated.

Definition 3.1 establishes that the Δ-differential systems denoted as S1, as given by (2.1) and (2.2), are deemed completely controllable under the circumstance that, for a given initial state Z(t0) = Z0 at time t0 and any specified final state Zf, there exists a finite time t1 > t0 and a control ˆU(t) for t0 ≤ t1 such that Z(t1) = Zf.

Theorem 3.2 presents the proposition that the time scale dynamical system S1 attains complete controllability within the closed interval J = [t0,t1] if and only if the n2 × n2 symmetric controllability matrix...

$$(3.1) \qquad V(t_0, t_1) = \int_{t_0}^{t_1} \phi(t_0, \sigma(s))(D \otimes C)(s)(D \otimes C)^*(s)\phi^*(t_0, \sigma(s))\Delta s,$$

where $\phi(t, s)$ *is defined in* (2.4), *is non-singular. In this case the control*

$$(3.2) \qquad \hat{U}(t) = -(D \otimes C)^*(t)\phi^*(t_0, \sigma(s))V^{-1}(t_0, t_1)\{Z_0 - \phi(t_0, t_1)Z_f\}$$

Defined for the interval t0 ≤ t1, the transfer of Z(t0) = Z0 to Z(t1) = Zf is achieved by the operation of V(t0,t1), assuming its nonsingularity.

Proof. When V(t0,t1) is established as non-singular, the control outlined in (3.2) comes into existence. Subsequently, upon substituting (3.2) into (2.5) with t = t1, the resulting equation is obtained.

$$Z(t_1) = \phi(t_1, t_0)[Z_0 - \int_{t_0}^{t_1} \phi(t_0, \sigma(s))(D \otimes C)(s)(D \otimes C)^*(s)\phi^*(t_0, \sigma(s))$$
$$\times V^{-1}(t_0, t_1)\{Z_0 - \phi(t_0, t_1)Z_f\}\Delta s] = Z_f.$$

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Hence the dynamical system S_1 is completely controllable.

Conversely, suppose that the dynamical system S_1 is completely controllable on J, then we have to show that $V(t_0, t_1)$ is non singular. Then there exists a non zero $n^2 \times 1$ vector α such that

$$\alpha^* V(t_0, t_1)\alpha = \int_{t_0}^{t_1} \alpha^* \phi(t_0, \sigma(s))(D \otimes C)(s)(D \otimes C)^*(s)\phi^*(t_0, \sigma(s))\alpha\Delta s$$

$$= \int_{t_0}^{t_1} \theta^*(\sigma(s), t_0)\theta(\sigma(s), t_0)\Delta s$$

$$\alpha^* V(t_0, t_1)\alpha = \int_{t_0}^{t_1} \|\theta\|^2 \Delta s \geq 0 \tag{3.3}$$

, where $\theta = (D \otimes C)^*(s)\phi^*(t_0, \sigma(s))\alpha$. From (3.3) $V(t_0, t_1)$ is positive semi definite. Suppose that there exists some $\beta \neq \mathbf{0}$ (zero vector) such that $\beta^* V(t_0, t_1)\beta = 0$, then from (3.3) with $\theta = \eta$, when $\alpha = \beta$ implies

$$\int_{t_0}^{t_1} \|\eta\|^2 \Delta s = 0.$$

Using the properties of norm, we have

$$\eta(\sigma(s), t_0) = \mathbf{0}, \quad t_0 \leq t \leq t_1. \tag{3.4}$$

Since S_1 is completely controllable, so there exists a control $\hat{U}(t)$ making $Z(t_1) = \mathbf{0}$ if $Z(t_0) = \beta$. Hence from (2.5), we have

$$\beta = -\int_{t_0}^{t_1} \phi(t_0, \sigma(s))(D \otimes C)(s)\hat{U}(s)\Delta s.$$

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