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A Study on Random Maximum Degree Based Vertex Graceful Labeling Graph and Generalized Gone Graphs

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Abstract:

Elsonbaty and Daoud [19,22] introduced a new type of labeling called Edge Even Graceful Labeling. He also stated a necessary and sufficient condition for Even Edge Labeling of Path Graphs, Cycle Graphs, Star Graphs, Friendship Graphs, Wheel Graphs, Double Wheel Graphs, and Fan Graphs. In this chapter, we described and studied the concept of Maximum Degree Based Vertex Graceful Labeling Graph with Even labeling on edges. We characterized and illustrated the concept of Maximum Degree Based Vertex Graceful Labeling with Even labeling on Edges for $\mathcal{PG}, \mathcal{CG}, S_n$ Graph, $\mathcal{PRG}$ and closed S_n graph, exg, and $\mathcal{M}vg$ of $\mathcal{CG}$.

Keywords: Maximum random degree graph, general cone graph. graceful labeling graph, cycle graph

Introduction:

A Graph G with p vertices and q edges is said to be Maximum Degree Based Vertex Graceful Labeling Graph (MCDBVG - ELE Graph) with Even Labeling on Edge if there exists a bijection f from the edge set to the set $\{2,4,6,\dots,2q\}$ so that the induced mapping the vertex set to the set $\{0,1,2,\dots(2q-1)\}$ which is given by $f^*(u) = \left\{\left[\frac{\sum f(uv)}{\Delta}\right] / uv \in E(G)\right\}$, where Δ is maximum degree of G. [] denotes an integral part.

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Generalized Gone Graphs

In This research paper, we presented the gracefulness of some graph classes and how to construct bigger graceful graphs from smaller ones. In this chapter, we generalize the wheel graphs, also known as cone graphs, and study its gracefulness. This graph class was first studied by Bhat-Nayak and Selvam [6] in 2003 and not much progress has been made since then.

A generalized cone graph is the join of a cycle graph C_p and an independent set I_q, where $p \geq 3$ and $q \geq 0$. For instance, for $q = 0$ and $q = 1$, we simply have the cycle graphs and the wheel graphs, respectively.

Throughout this chapter, we denote the vertices of the generalized cone graphs as $V\left(C_p + I_q\right) = \left\{u_0, u_1, \ldots, u_{p-1}, v_0, v_1, \ldots, v_{q-1}\right\}$ where $u_k \in V\left(C_p\right), u_k u_{k+1} \in E\left(C_p\right)$ for $0 \leq k < p$ and $u_p = u_0$, and $v_k \in V\left(I_q\right)$. Also, from now on, we simply call generalized cone graphs as cone graphs.

The first result we show is concerning the non-graceful cone graphs. As we said in Chapter 2, the only useful theoretical tool for proving the non-existence of graceful labeling for a given graph is the parity condition, which only applies to Eulerian graphs. Thus, applying the parity condition to Eulerian cone graphs, the following holds.

Proposition 1.1. The cone graph $C_p + I_q$ is not graceful for $p \equiv 2(\mathrm{mod}4)$ and $q \equiv 0(\mathrm{mod}2)$

Proof. For $p \equiv 2(\mathrm{mod}4)$ and $q \equiv 0(\mathrm{mod}2)$, the cone $C_p + I_q$ is Eulerian since the degree of every vertex is even (cf. |7|), and it has $m = p(q+1)$ edges. Writing $p = 4s + 2$ and $q = 2t$, we have $m = (4s+2)(2t+1) \equiv 2(\mathrm{mod}4)$. Hence, by the parity condition, $C_p + I_q$ is not graceful.

1.1 Graceful cones

For $q = 0$ and $q = 1$, we have the cycle graphs and the wheel graphis, respectively, and their gracefulness is already characterized in Chapter 2 . For $q = 2$, we have the double cones, and it is still an open problem to characterize them. By Proposition 4.1, the double cone $C_p + I_2$ is not graceful for $p \equiv 2(\mathrm{mod}4)$, and so far they are the only non-graceful double cones [6,11,19].

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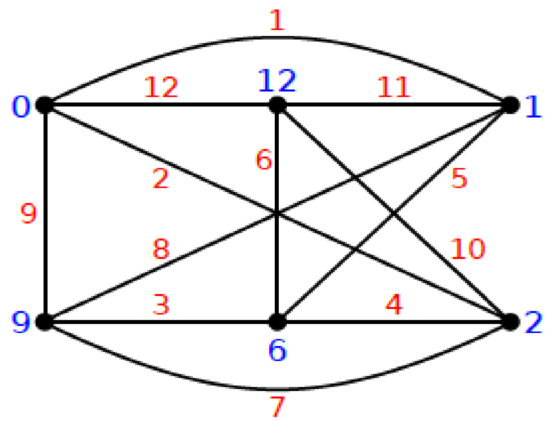

Figure 1.1: Graceful labeling of $C_4 + I_2$.

For the general case, Bhat-Nayak and Selvarn [6] proved the following theorem.

Proposition 1.2. The cone graph $C_p + I_q$ is graceful for $p \equiv 0,3 \pmod{12}$ and $q \geq 1$.

For the proof of Proposition 1.2, Bhat-Naysk and Selvam introduced a new graph labeling and showed s more general result similar to Theorem 2. 7 .

A vertex lsbeling f of a graph G with n vertices is said to be a special labeling if it satisfies the following conditions:

1 For every $i \in [1, n]$, there exists s vertex $u_i \in V(G)$ such that $f(u_1)$ is either $2i - 1$ or $2i$.

2 $\text{Im}\left(f_\gamma\right) = [1,2n] \setminus \text{Im}\,(f)$.

3 If $f(x)$ and $f_\gamma(xy)$ are odd, then $f(x) < f(y)$.

Note that conditions 1 and 2 imply that the number of vertices must be the same as the number of edges, i.e., $n = m$.

Theorem 1.3. If a graph G has a special labeling, then the graph $G + I_q$ is graceful for all $q \geq 1$.

Proof. Let G be s graph on p vertices and f be s special labeling of G. Define the vertex labeling g for $G + I_q$ ss follows, where $V(G) = \{u_1, \ldots, u_p\}$ and $V(I_q) = \{v_1, \ldots, v_q\}$:

$$g(v_j) = j - 1$$

$$g(u_i) = \begin{cases} i(q+1) & \text{if } f(u_i) = 2i \\ i(q+1) - 1 & \text{if } f(u_i) = 2i - 1 \end{cases}$$

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We claim g is a graceful labeling of $G + I_q$. As noted before, since G has a special labeling, G has p edges. Thus, the number of edges of $G + I_q$ is $p + pq$. Clearly, $g: V(G + I_q) \to [0, p(q+1)]$ and it is injective. So, we have to prove that g_γ is onto $[1, p(q+1)]$. For that, we show that for each $i \in [1,p]$ and $j \in [1, q+1]$, there is an edge e with $g_\gamma(e) = (i-1)(q+1) + j$.

Consider a pair (i,j). Since f is a special labeling of G, by condition 1, there is a vertex $u_i \in V(G)$ with $f(u_i) = 2i - 1$ or $f(u_1) = 2i$.

Case 1. $f(u_1) = 2i - 1$ and $1 \le j \le q$.

We have $g(u_i) = i(q+1) - 1$ and $g(v_{q-j+1}) = q - j$. Since $q - j < i(q+1) - 1$, the edge label on $u_1 v_{q-j+1}$ is $i(q+1) - 1 - (q-j) = (i-1)(q+1) + j$.

Case 2. $f(u_1) = 2i - 1$ and $j = q + 1$.

By condition 2, there is an edge $e = xy \in E(G)$ with $f_\gamma(xy) = 2i$. Hence, $f(x)$ and $f(y)$ have the same parity. Suppose $f(x) = 2a + r$ and $f(y) = 2b + r$, where $r \in \{0,1\}$ is the parity. Then, $f_\gamma(xy) = 2i = |(2a + r) - (2b + r)| = 2|a - b|$, and $i = |a - b|$. Therefore, $g_\gamma(xy) = |(a(q+1) - r) - (b(q+1) - r)| = (q+1)|a - b| = i(q+1) = (i-1)(q+1) + (q+1)$

Case 3. $f(u_1) = 2i$ and $2 \le j \le q + 1$.

We have $g(u_i) = i(q+1)$ and $g(v_{q-j+2}) = q - j + 1$. Since $q - j + 1 < i(q+1)$, the edge label on $u_i v_{q-j+2}$ is $i(q+1) - (q - j + 1) = (i-1)(q+1) + j$.

Case 4. $f(u_i) = 2i$ and $j = 1$.

By condition 2, there is an edge $e = xy \in E(G)$ with $f_\gamma(xy) = 2i - 1$. Now, $f(x)$ and $f(y)$ have different parities. Without loss of generality, suppose $f(x)$ odd and let $f(x) = 2a - 1$ and $f(y) = 2b$. By condition 3, we have $f(x) < f(y)$ which implies $g(x) < g(y)$. Thus, $f_\gamma(xy) = 2i - 1 = 2b - (2a - 1)$ implies $i - 1 = b - a$. Finslly, $g_\gamma(xy) = b(q+1) - (a(q+1) - 1) = (b-a)(q+1) - 1 = (i-1)(q+1) - 1$.

Thus, we have proved that In $(g_y) = [1, p(q+1)]$ and therefore g is a graceful labeling of $G + I_q$.

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We do not present here the complete proof of Proposition 4.2. Here, we only show a partial result which says that $C_2k_k + I_q$ is graceful. For that, Bhat-Nayak and Selvam proved the following lemmas.

Lemma 1.4. For $k \geq 2, P_{4k-3}$ has a vertex labeling f such that $\text{Im}\,(f) = [k+2,2k] \cup [2k+3,3k+1] \cup [5k+1,7k-1], \text{Im}\,(f_\gamma) = [2k+1,6k-4]$, and the end vertices reoive the labels $5k+1$ and $7k-1$.

Proof. Let $P_{4k-3} = u_1u_2 \cdots u_{4k-3}$ and define the vertex labeling f as follows:

$$
\begin{aligned}
f(u_{2i-1}) &= 5k+i && \text{for } 1 \leq i \leq 2k-1 \\
f(u_{2i}) &= k+2 && \text{for } i = 1 \\
&= 3k+3-i && \text{for } 2 \leq i \leq k \\
&= 3k+1-i && \text{for } k+1 \leq i \leq 2k-2
\end{aligned}
$$

Now, it is easy to verify directly that $\text{Im}\,(f_\gamma) = [2k+1,6k-4]$.

Remark 1.1. For $k = 1$, consider the single vertex of P_1 labeled with 6 .

Lemma 1.5. For $k \geq 1, P_{8k-1}$ has a vertex labeling f such that $\text{Im}\,(f) = [1,k] \cup [k+2,8k], \text{Im}\,(f_\gamma) = [1,8k-2]$, and the end vertices receive the labels $2k+1$ and $8k$.

Proof. Let $P_{8k-1} = u_1u_2 \cdots u_{8k-1}$ and define the vertex labeling f as follows:

$$
\begin{aligned}
f(u_1) &= 2k+1 && \\
f(u_{2t+1}) &= 4k+1+i && \text{for } 1 \leq i \leq k \\
f(u_{2t}) &= 4k+2-i && \text{for } 1 \leq i \leq k \\
f(u_{8k+1-2u}) &= 8k+1-i && \text{for } 1 \leq i \leq k+2 \\
f(u_{\varepsilon k-2}) &= 2k+2 && \\
f(u_{8k-2-2u}) &= i && \text{for } 1 \leq i \leq k
\end{aligned}
$$

Thus, we labeled the vertices $u_1, \ldots, u_{2k+1}, u_{6k-3}, \ldots, u_{8k-1}$ with labels in $[1,k] \cup [2k+1,2k+2] \cup [3k+2,5k+1] \cup [7k-1,8k]$, and obtained edge labels in $[1,2k] \cup [6k-3,8k-2]$. For the remaining subpath $u_{2k+1}u_{2k+2} \cdots u_{6k-3}$, label it as given by

Lemma 1.4 to obtain the desired labeling.

Lemma 1.6. For $k \geq 1, P_{8k-1}$ has a vertex labeling g such that $\text{Im}\,(g) = \{16k+2,16k+4,\ldots,18k\} \cup \{18k+4,18k+6,\ldots,32k\}, \text{Im}\,(f_\gamma) = \{2,4,\ldots,16k-4\}$, and the end vertices receive the labels $20k+2$ and $32k$.

Proof. Let f be the vertex labeling obtained from Lemms 4.5. Then, defining g as $g(u) = 2f(u) + 16k$ gives the required labeling.

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Lemma 1.7. For $k \geq 1$, P16k+3 has a vertex labeling f such that $\text{Im}\,(f) = \{1,3, \ldots,, 16k-1, 18m+2, 20k+2, 32k, 32k+2, \ldots, 48k\}, \text{Im}\,(f_\gamma) = \{16k-2, 16k, 16k+1,, 16k+3, \ldots, 48k-1\}$, and the end vertices receive the labels $20k+2$ and $32k$.

Proof. Let $P_{16k+3} = u_1 u_2 \cdots u_{16k+3}$ and define the vertex labeling f as follows:

$$\begin{aligned} f(u_{2u-1}) &= 20k+2 && \text{for } i = 1 \\ &= 48k+4-2i && \text{for } 2 \leq i \leq 8k+2 \\ f(u_{2i}) &= 2i-1 && \text{for } 1 \leq i \leq 7k \\ &= 18k+2 && \text{for } i = 7k+1 \\ &= 2i-3 && \text{for } 7k+2 \leq i \leq 8k+1 \end{aligned}$$

Now, it is easy to verify that $\text{Im}\,(f_\gamma)$ is as required.

Proposition 1.8. The cone graph $C_{24k} + I_q$ is graceful for all $k \geq 1$. Proof. Consider P_{8k-1} and P_{16k+3} labeled as given by Lemmas 4.6 and 4.7 respectively. By joining the paths by identifying the end vertices with the same label, we get a C_{24k} with a vertex labeling f such that $\text{Im}\,(f) = \{1,3, \ldots, 16k-1, 16k+2, 16k+4, \ldots, 48k\}$ and $\text{Im}\,(f_\gamma) = \{2,4, \ldots, 16k, 16k+1, 16k+3, \ldots, 48k-1\}$. Furthermore, the largest odd vertex label is less than the smsllest even vertex label. Therefore, f satisfies all three conditions of being a special lsbeling for C_{24k}.

Therefore, by Theorem 1.3, $C_{24k} + I_q$ is graceful.

For the proof of Proposition 4.2, Bhat-Nayak and Selvam proved not only **Proposition 1.8**, but also that $C_p + I_q$ is graceful for $p \equiv 3,12,15 \pmod{12}$, ench of them following the same strategy as shown before: prove the existence of a specific vertex labeling of some specific paths and then join their end vertices to form a cycle graph.

Besides Proposition 1.2, Bhat-Nayak and Selvam also proved the following proposition.

Proposition 1.9. The cone graph $C_p + I_q$ is graceful for $p = 7,11,19$ and $q \geq 1$.

Proof. The following vertex labeling are special labeling for their respective cycle.

C_7: 1,14,5,7,10,4,12.

C_{11}: 1,22,5,18,7,15,9,12,14,4,20.

C_{19}: 1,36,3,34,5,32,7,30,12,26,16,22,20,24,13,28,9,17,38.

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Brundage |8| also worked on this problem and showed the following result. Brundage [8] organized the gracefulness of cone graphs in a table (see Table 4.1) and made a conjecture characterizing this class.

Conjecture 1.1 (Brundage, 1994). The generalized cone graph $C_p + I_q$ is graceful if, and only if, the parity condition holds.

Table 1.1: Gracefulness of $C_p + I_q$ (updated as of 2014).

Conjecture 1.1 (Brundage, 1994). The generalized cone graph $C_p + I_q$ is graceful if, and only if, the parity condition holds.

p	3,4	5	6	7,8	9	10	11,12	13	14	comments
0	Y	N	N	Y	N	N	Y	N	N	Y iff $p \equiv 0,3(\mathrm{mod}4)$
1	Y	Y	Y	Y	Y	Y	Y	Y	Y	Y$\forall p$
2	Y	Y	N	Y	Y	N	Y	?	N	? , N$\forall p = 6 + 4k$
3	Y	Y	Y	Y	Y	?	Y	?	?	?
4	Y	Y	N	Y	Y	N	Y	?	N	? , N$\forall p = 6 + 4k$
5	Y	Y	?	Y	?	?	Y	?	?	?
6	Y	Y	N	Y	?	N	Y	?	N	? , N$\forall p = 6 + 4k$
7	Y	Y	?	Y	?	?	Y	?	?	?

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8	Y	Y	N	Y	?	N	Y	?	N	? , N$\forall p$ $= 6 + 4k$
9	Y	Y	?	Y	?	?	Y	?	?	?
comments	Y	Y	?,,			?,,			?,,	?,,

Table 1.1: Gracefulness of $C_p + I_q$ (updated as of 2014).

1.2 Computational results

Questioning the validity of Conjecture 4.1, we started looking for counterexamples, i.e., find a cone graph for which the parity condition does not hold and it is not graceful. For this task, a backtracking search algorithm similar to the Fang's algorithm presented in this research paper was implemented.

The strategy is the same as in Fang's algorithm: it tries to create a new edge label at each iteration by labeling a not yet labeled vertex. For reducing the search tree, some optimizations were made due to the inherent symmetries of cone graphs. The following observations eliminate most of search through equivalent labeling given by the symmetries of the graph.

The function Check in the pseudocode checks if the current labeling is valid, i.e., it checks if there are no repeated edge or vertex labels. Unlike Fang's backtracking search algorithm, this check is necessary here because labeling a vertex can create more than just one edge label. So, a verification is necessary every time we label a new vertex before continuing the search.

Running the search for a graceful labeling for $C_6 + I_5$, the smallest cone graph which was still unknown to be graceful or not, the algorithm returned no possible graceful labeling, refuting, therefore, Brundage's conjecture. Moreover, the algorithm did not find a graceful labeling for $C_6 + I_q$ with $5 \leq q \leq 35$. Notice that we are only interested in odd values of q since, for even values, the parity condition already settles that $C_6 + I_q$ is not graceful.

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Searching for more non-graceful cone graphs, it makes sense to look for cone graphs $C_p + I_q$ with $p \equiv 2(\text{mod}4)$ as they are the only ones that, together with an even q, are not graceful by the parity condition. Then, the next subclass to search for non-graceful cones is $C_{10} + I_q$. We found that $C_{10} + I_3$ and $C_{10} + I_5$ are graceful. However, the algorithm returned no graceful labeling for $C_{10} + I_q$ with $7 \leq q \leq 25$. A similar result was gotten with $p = 14$: the cones $C_{14} + I_3$ and $C_{14} + I_5$ are graceful, but the cones $C_{14} + I_7$ and $C_{14} + I_9$ are not. The following propositions summarize these results.

$C_{10} + I_3$: 0,40,25,3,33,13,6,29,10,21; 37,38,39.
$C_{10} + I_5$: 0,27,1,57,14,13,2,16,3,15; 23,32,51,55,60.
$C_{14} + I_3$: 0,56,6,1,28,5,2,30,34,3,33,11,22,55; 40,47,54.
$C_{14} + I_5$: 0,84,33,17,82,34,47,54,64,68,69,32,49,83; 2,5,8,11,14.

Proposition 1.12. The cone graphs $C_6 + I_q, 5 \leq q \leq 35, C_{10} + I_q, 7 \leq q \leq 25$, $C_{14} + I_7$, and $C_{14} + I_9$ are not graceful

Proof. Proven computationally.

Proposition 1.12 not only disproves Conjecture 4.1, but also gives a stronger feeling about how the non-gracefulness of generalized cone graphs behaves, from which we conjecture the following.

Conjecture 1.2. For every $p \equiv 2(\text{mod}4)$, there erists a $q_p > 1$ such that the cone graph $C_p + I_q$ is not goaceful for all $q \geq q_p$.

One might think of trying to prove it computationally, implementing an algorithm to do something similar to the proof of Proposition 2.2, exhausting all possibilities for all values of q greater than a threshold. However, as it was noted,

the running times of the algorithm to establish the non-gracefulness were growing exponentially, which indicates that it is not possible to prove it in this way.

Besides the non-graceful cone graphs, we also searched for new farnilies of cones which are graceful. We have seen two approaches to tackle this class: fixing the size of the independent size or fixing the size of the cycle. By taking the last one, we started to find graceful labeling for $C_9 + I_q$, the smallest family of this kind which was still open, and tried to find a pattern in the labeling while increasing the size of the independent size. As seen in Proposition 4. 10, a simple rule could be possible, and indeed we found a scheme of labeling, not only for $C_9 + I_q$, but also for $C_{13} + I_q$.

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$$f(v_k) = \begin{cases} 1 & \text{if } k = 0 \\ 9k + 4 & \text{if } k = 1,2,\dots,q-1 \end{cases}$$

For $C_{13} + I_q$, label the vertices of C_{13} ass $0, m, m-8, 6, m-9, 10, m-6, 7, m-4, m-7, 5, m-1, m-3$ along the cycle, where $m = 13(q+1)$ is the number of edges, and label I_q se follows:

$$f(v_k) = \begin{cases} 1 & \text{if } k = 0 \\ 13k + 4 & \text{if } k = 1,2,\dots,q-1 \end{cases}$$

As for the verification, since it is analogous to the proof of Proposition 4.10, it is omitted.

On the other hand, finding a pattern after having fixed the size of the independent set (and allowing the size of the cycle to grow freely) seems to be much harder. For instance, it seems that, for $p > 5$ and $p \equiv 1 \pmod 4$, the cone graph $C_p + I_q$ has a graceful labeling f such that $f(v_0) = 1$ and $f(v_k) = pk + 4$ for $1 \leq k < q$, as it can be seen in

Proposition 1.13; we have also verified it for several cones with $p = 17$ and $p = 21$. However, no pattern has been found for the cycles. Another example is the family of cone graphis $C_p + I_q$ with $p \equiv 0 \pmod 4$: each of them seems to have a graceful labeling with $f(v_k) = \frac{p}{4}(k+1)$ for $0 \leq k < q$. That is known to be true for $p = 4|6|$, and now $p = 8$ (see Remark 4.2); we have also verified that several cones with $p = 12,16,20$ admit such labeling.

Table 1.2 summarizes the current state of the gracefulness of generalized cone graphs for small values and gives a comment for the state of each row and column.

Conclusion:

The graceful labeling of graphs has been a topic of research for 50 years and it still has many properties to be found. Although its primary interest was the graceful labeling of trees in order to solve Ringel's conjecture, graceful labeling of graphs gained over the years its own beauty and interest. This work gives a brief overview of the subject, presenting not only theoretical results from the literature, but also some computational results. Furthermore, we give some contributions to this problem.

In this research paper, we move our focus to generalized cone graphs. Their gracefulness was first tackled by Bhat-Nayak and Selvam, although some particular cases

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were already known. Later, Brundage also worked on this graph class and made a conjecture characterizing the gracefulness of cone graphs. We tackled the gracefulness of cone graphs computationally and were able to disprove Brundage's conjecture. We also establish the gracefulness of new families of cone graphs and make a new conjecture regarding the non-graceful cone graphs.

For future work on the subject, we could consider looking for a way to prove

Conjecture 1.2, or even characterize the gracefulness of generalized cone graphs. As we showed in Chapter 4, it seems that $Cp + Iq$ is graceful for p _ 0; 1; 3 (mod 4) and q _ 1. For p _ 2 (mod 4), our conjecture says there is a $qp > 1$ such that the cone graph is not graceful for all q _ qp. If, moreover, we could find out the parameter qp for each p _ 2 (mod 4), we would have a characterization of the gracefulness of generalized cone graphs.

Another class of interest is the class of trees, being the main open class on this topic. It is already settled that many classes of trees are graceful, but also there are many classes, even simple ones like lobsters, that are still open. Finally, another approach to the problem is to relax the conditions of graceful labeling and find nearly graceful labeling. This approach by approximating the labeling is also a topic of research for both trees and graphs in general.

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