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MAXWELL'S EQUATIONS AND YANG-MILLS EQUATIONS IN COMPLEX VARIABLES: NEW PERSPECTIVES

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Abstract

Starting from Maxwell's theory of electromagnetism in a Minkowski spacetime, we generalize to arbitrary spacetimes and gauge groups. The gauge groups U(1) and SU(3) and their associated Yang-Mills theories are discussed in detail. This paper provides a view of Maxwell's equations from the perspective of complex variables. The study is made through complex differential forms and the Hodge star operator in C2 with respect to the Euclidean and the Minkowski metrics. It shows that holomorphic functions give rise to nontrivial solutions, and the inner product between the electric and the magnetic fields is considered in this case. Further, it obtains a simple necessary and sufficient condition regarding harmonic solutions to the equations. In the end, the paper gives an interpretation of the Lorenz gauge condition in terms of the codifferential operator.

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1 Introduction

Arguably the most fundamental pursuit of theoretical physics is that of unification of the laws of nature. The development of special relativity by Einstein in the early 20th century and the later development of Yang-Mills theory both have origins in Maxwell's theory of electromagnetism. Naturally then, we begin by developing some of the formalism of special relativity and Maxwell's theory of electromagnetism. Then, we develop some tools from diﬀ erential geometry to generalize the vector calculus used in the description of Maxwell's theory. This development will in turn al- low us to understand what is considered the cornerstone of modern theoretical physics, Yang-Mills theory, which more or less gives us a prescription for developing theories of the behavior of matter all around us (besides behavior that is due to gravity). Particularly we will discuss the theories of Quantum Electrodynamics (QED), which describes how electrically charged particles and photons interact, and Quantum Chromodynamics, which describes how the nuclei of atoms are formed and behave. As we will see, these particular Yang-Mills theories are U(1) and SU(3) gauge invariant, respectively, and in some sense the only diﬀ erence between them are described by the properties ofthese groups!

We will find many pieces of notation useful, but not by any means universal, so we shall clarify maybe the most ubiquitous of these now, since the following we will begin using it immediately. We will work in what some call "god-given" units, where the speed of light, $c \approx 3 \times 10^8$ m/s and the reduced Planck's constant, $\hbar \approx 4 \times 10^{-15}$ eV·s are set to unity. This will make Maxwell's equations, Lorentz transforms, and the wave equations of quantum mechanics much less cluttered. For example, Einstein's famous mass-energy equivalence, $E = mc^2$ now reduces to

$$E = m. \tag{1.1}$$

In practice this notation makes calculations less tedious, and in order to get back the desired units of some quantity, one simply multiplies by the correct (unique) factor of c's and $\hbar$'s. We will not really discuss the nuances of units after this point, but it is worth mentioning since some readers may be unfamiliar with this somewhat confusing (albeit convenient) practice. When it makes sense, we will keep these units around to make concepts more transparent.

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2 Maxwell's Equations and Special Relativity

Special Relativity

Special relativity is concerned with how measures of space and time differ from one inertial reference frame to another (that is, two frames moving at constant velocities). The fundamental quantities in relativity are four component vectors (or 4-vectors), like space-time, denoted using a Greek index that runs from 0 to 3, (e.g., $\mu = 0, 1, 2, 3$). The space-time vector is defined as $x^\mu = (t, x, y, z)^T = (t, \vec{r})^T$, in terms of indices,

$$x^0 = t, \quad x^1 = x, \quad x^2 = y, \quad x^3 = z. \tag{2.1}$$

Specific points in spacetime $x^\mu \in \mathbb{R}^4$, are called *events*. Particles follow continuous trajectories of events called *world lines*. We will be concerned with events and world lines as they are seen by two different inertial reference frames, O and O'. As we will see shortly, Minkowski spacetime is a manifold, specifically, a manifold with a Lorentz metric

$$\eta_{\mu\nu} = \eta^{\mu\nu} \doteq \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.2}$$

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Using this metric we can lower the index of our space-time vector in the following way

$$x_\mu = \sum_{\mu=0}^{3} \eta_{\mu\nu} x^\nu = (t, -\vec{r}). \qquad (2.3)$$

Following Einstein's summation convention, we will suppress the summation notation whenever wesee a repeated index in an expression, so the above becomes (3.5)

$$x_\mu = \eta_{\mu\nu} x^\nu = (t, -\vec{r}). \qquad (2.4)$$

Vector Fields

As far as we are interested here, a vector field will be a *diff erential operator* defined at each point on a manifold whose sole ambition in life is to diff erentiate smooth functions. We will start by naming a few things. The set of smooth real valued functions on a manifold M is denoted $C^\infty(M)$, and is a *commutative algebra* over the real numbers. Formally, this property means the following:

let f, g, h ∈ $C^\infty(M)$ and α, β ∈ R. Then at each point in M we have

$$f + g = g + f$$

$$f + (g + h) = (f + g) + hf(gh) = (fg)h$$

$$f(g + h) = fg + fh(f + g)h = fh + gh1f = f$$

$$\alpha(\beta f) = (\alpha\beta)f\, \alpha(f + g) = \alpha f + \alpha g(\alpha + \beta)f = \alpha f + \beta f$$

$$fg = gf$$

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All a vector field does, in some sense, is take one function on a manifold to another function of the same manifold. Abstractly, we say $v : C^{\infty}(M) \to C^{\infty}(M)$, where v is a vector field. But before we define vector fields on a manifold, we will consider the familiar case in R^n. The directional derivative of a function $f : R^n \to R$ in the direction of the vector field v is written

$$vf = v^{\mu}\partial_{\mu}f \tag{3.6}$$

where $\mu = 1, 2, ..., n$ (in general, but we will have $n = 4$ for our purposes). Actually, this formulawill hold for all $f \in C^{\infty}(R^n)$, so we can just write.

$$v = v^{\mu}\partial_{\mu} \tag{3.7}$$

A *vector field* on M will have the same basic properties as a diﬀ erential operator in R^n. Namely, linearity and the Leibniz law define a vector field on M.

Definition 3.3. Let v be a vector field on M, $\alpha \in R$, and $f, g \in C^{\infty}(M)$. Then we have:

$$v(f + g) = vf + vg$$

$$v(\alpha f) = \alpha v(f)$$

$$v(fg) = (vf)g + f(vg),$$

(3.8) where the first two are simply linearity, and the last is the Leibniz law or product rule. We should also note that the components of the vector field, v^{μ}, can themselves be functions of M. We now let Vec(M) be the set of all vector fields on M. Then Vec(M) is a module over $C^{\infty}(M)$, that is,for $f, g \in C^{\infty}(M)$ and $v, w \in Vec(M)$ we have

$$f(v + w) = fv + fw$$

$$(f + g)v = fv + gv$$

$$(fg)v = f(gv)1v = v,$$

where “1” is the constant function equal to $1 \in R$ on all of M.

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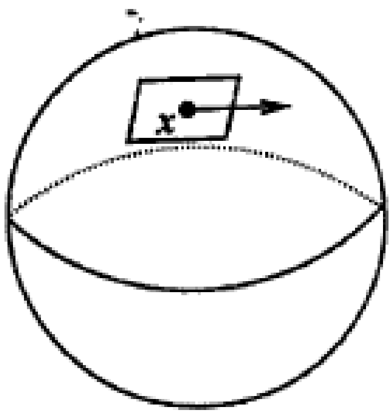

(3.9)

Figure 1: S^2 with a tangent vector at the point $x \in S^2$ [1]

Tangent Vectors

A convenient way to imagine a vector field on M is simply as an arrow assigned to each pointp $\in$ M. This arrow is called a *tangent vector* at the point p, and is denoted v_p. Where do these tangent vectors live? They belong to the set of all tangent vectors at p, called the *tangent space* at p, denoted $T_p(M)$. We should note, however, that unlike the vector space R^n, it only makes sense to add and subtract vectors that are at the same point on a manifold. The reason we can get away with adding and subtracting vectors at diﬀ erent points in R^n is because each tangent space has the same basis.

What, then, is "d" really doing here? We start first by recalling the gradient in R^n, ∇f, which wecan think of as an instruction to take $v \in Vec(R^n)$ to the directional derivative vf as

$$\nabla f(v) = vf. \tag{3.17}$$

The essential properties of the map are, once again, linearity and the Leibniz law. That is, if

$f, g \in C^\infty(R^n)$ and $v, w \in Vec(R^n)$, we have

$$\nabla f(v + w) = \nabla f(v) + \nabla f(w)$$

$$\nabla f(gv) = g\nabla f(v)$$

$$\nabla (fg)(v) = g\nabla f(v) + f\nabla g(v)$$

or in other words, the gradient $\nabla f : \mathrm{Vec}(R^n) \to C^\infty(R^n)$ is linear over $C^\infty(R^n)$. (3.18)
We now define the differential forms on M, using only the above three properties to define ourgeneralized gradient. Here it is:

$$df(v) = vf. \tag{3.19}$$

The 1-form df is called the *differential* of f, or the *exterior derivative* of f. So what is a 1-form on R^n? It is what you obtain when attempting to make (3.17) and (3.19), the gradient multiplied by the basis of differentials in R^n, $\{dx^\mu\}$:

$$df = \partial_\mu f dx^\mu. \tag{3.20}$$

5 Quantized Yang-Mills Theories

Strictly speaking, what we have developed in the last chapter is a way to write physical theories on arbitrary spacetimes. The applications of physical theories on curved spacetimes are generally either astrophysical in nature, or Beyond the Standard Model (BSM) of particle physics. An ex- ample may be helpful. One object of study that requires Maxwell's equations in curved space are magnetars [11], which are extremely dense remnants of supernovae that have a very strong mag- netic field (about 10^{10} times stronger than any magnet on Earth).

What we have developed so far is considered to be a classical Yang-Mills theory. In order to develop physical theories, we need to quantize the Yang-Mills field. Proper quantization of a field is a very involved effort, so rather than actually quantizing a field, we will sketch how to do thisand interpret the results.

Lagrangian Mechanics

In classical physics, there is a coordinate-free generalization of Newtonian mechanics called *Lagrangian Mechanics* which relies on knowing the quantity

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$$L = T - V, \tag{5.1}$$

called the *Lagrangian* of a system of particles, where T is the kinetic energy and V is the potential energy. The Lagrangian of a system of N particles in R^3 will then be described by 3N coordinates

$\{q_i(t) : i = 1, ..., 3N\}$ and 3N velocities $\{\dot{q}_i(t) = \frac{dq_i}{dt} : i = 1, ..., 3N\}$, where q_i and $\dot{q}_i$ are treated as independent variables. Given the Lagrangian of a physical system, we can determine the pathsthat the particles take by minimizing the *action* [8]

$$S = \int_{t_1}^{t_2} L(q_i, \dot{q}_i)dt. \tag{5.2}$$

We say an action is minimized when the variation $\delta S = 0$ to linear order in $\delta q_i(t)$ under thetransformation

$$\delta q_i(t) \longrightarrow q_i(t) + \delta q_i(t) \tag{5.3}$$

where $\delta q_i(t)$ is a smooth function that is zero at the limits of integration, i.e.

$$\delta q_i(t_1) = \delta q_i(t_2) = 0. \tag{5.4}$$

The variation of the action is

$$\delta S = \int_{t_1}^{t2} \left(\frac{\partial L}{\partial q_i}\delta q_i + \delta \dot{q}_i \frac{\partial L}{\partial \dot{q}_i} \right) dt \tag{5.5}$$

$$= \int_{t_1}^{t2} \delta q_i \left[\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} \right] dt + \left. \delta q_i \frac{\partial L}{\partial \dot{q}_i} \right|_{t_1}^{t2}$$

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where we have performed integration by parts on the second term of the integrand. The term outside the integrand is 0 by (5.4), and since the function $\delta q_i(t)$ is arbitrary we have

$$\frac{\partial L}{\partial q_i} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = 0. \quad (5.6)$$

Hence, the action is minimized when (5.6), known as the *Euler-Lagrange equation*, is satisfied.

For relativistic fields[5], we cannot treat time as an entity distinct from spacial dimensions. In other words a field $\phi(x^\mu)$ is a function of spacetime. Instead of working with a Lagrangian, we will now work with a Lagrangian density

$$L = \int_V \mathcal{L}\, d^3x, \quad (5.7)$$

and make the following substitutions in the Euler-Lagrange equation

$$L \longrightarrow \mathcal{L},$$
$$q_i \longrightarrow \phi_i, \quad (5.8)$$
$$\frac{d}{dt} \longrightarrow \partial_\mu,$$

we can obtain the relativistic Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \partial_\mu\left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)}\right) = 0 \quad (5.9)$$

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for N fields. Though this equation is relativistically invariant, like the electromagnetic field, it is still a classical field. In order to obtain a quantum field, it needs to be "quantized", which formally amounts to promoting the dynamical variables (fields) to operators and imposing a canonical com- mutation relation [4].

The first field that we will look at is the Dirac field, which is a fermionic field whose excitations are spin 1/2 particles like electrons (e), as well as their heavier cousins the μ and τ , but also quarks, which make up particles like protons and neutrons. Fermionic fields are perhaps the most fundamental in all of quantum field theory; as a matter of fact, all physical matter (at least the kind we are familiar with on Earth) is composed of fermions.

The Dirac Equation

In the early 20th century, physicists sought a quantum mechanical wave equation that was com- patible with special relativity. To obtain the non-relativistic Schrödinger equation, we start with the classical energy momentum relation for a single particle

$$E = T + V = \frac{p^2}{2m} + V \tag{5.10}$$

and apply the "quantum prescription", which amounts to the substitutions

$$p \to i\nabla \qquad E \to i\frac{\partial}{\partial t} \tag{5.11}$$

to obtain the operator

$$i\frac{\partial}{\partial t} = -\frac{1}{2m}\nabla^2 + V. \tag{5.12}$$

The equation obtained by applying this operator to a function

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$$-\frac{1}{2m}\nabla^2 \Psi + V \Psi = i\frac{\partial}{\partial t}\Psi. \quad (5.13)$$

yields the Schrödinger equation, where Ψ is called the wave function, and is a function of position and time (which are treated as independent). The integral of the square of the wave function is set to unity

$$\int_{\text{All space}} \Psi^*\Psi \, d^3x = 1 \quad (5.14)$$

and we can find the probability that a particle will be in some region of space by simply integrating

over that region. As the wave function evolves with time the integral remains unitary. Following the same procedure with the relativistic energy momentum relation

$$p^\mu p_\mu - m^2c^2 = 0 \quad (5.15)$$

yields an equation that is second order in time. This turns out to be a problem, because a wave equation constructed using (5.15) will not remain unitary as time goes on. Schrödinger actually tried the second procedure before the first, and gave up when it failed to predict the hydrogen emission spectrum. Dirac later realized that an equation that was first order in time could be obtained by factoring the energy momentum relation,

Quantum Electrodynamics

Writing down the Dirac Lagrangian[6] only takes 3 more strokes of a pencil than the Dirac equation,

$$L_{Dirac} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi \quad (5.23)$$

but we can get considerably more mileage from a Lagrangian than a wave equation, primarily for two reasons. The first is that we can tell the symmetries of a theory from the Lagrangian as a consequence of Noether's theorem, which says that every conservation law (of energy, momentum, charge, etc.) correspond to a continuous symmetry of a Lagrangian. The second is that we can read the Feynman rules for calculating diff erent processes directly off of the Lagrangian.

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Together with the Maxwell Lagrangian,

$$\mathcal{L}_{Maxwell} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.24}$$

we almost have all of the ingredients of the QED Lagrangian,

$$\mathcal{L}_{QED} = \mathcal{L}_{Dirac} + \mathcal{L}_{Maxwell} + \mathcal{L}_{int} \tag{5.25}$$

where the last term, $\mathcal{L}_{int}$ is a yet unspecified interaction term. Without it we have a Lagrangian for a theory of two fields minding their own business. The interaction term can be determined by replacing ∂_μ with the gauge covariant derivative [4]

$$D_\mu = \partial_\mu + ieA_\mu \tag{5.26}$$

and seeing what "extra" term we pick up, where e is the coupling constant (the electrical charge) between the Dirac and electromagnetic field. Notice that if e = 0, the fields "decouple", or go back to minding their own business. Actually, now we can just write the QED Lagrangian as

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} \tag{5.27}$$

and see what extra term the gauge covariant produces when compared to (5.23) and (5.24) alone. When we work this out, the interaction term is

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu. \tag{5.28}$$

The theory of Quantum Electrodynamics is in some sense "solved". This claim is due to the fact that a quantitative description of all electromagnetic processes can be calculated to an incredible degree of precision by only evaluating a handful of integrals that can be read off from

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the Lagrangian. More accurately, these integrals[7] are terms in a series expansion of the scattering matrix, S: [4]

$$S = \sum_{n=0}^{\infty} S^{(n)} \tag{5.29}$$

with each term

$$S^{(n)} \propto \left(\frac{e^2}{4\pi}\right)^n \approx \left(\frac{1}{137}\right)^n . \tag{5.30}$$

Since the coupling constant e is relatively small, we really only have to evaluate the first terms in this series to get an accurate approximation of S. For the theory of strong nuclear force (Quantum Chromodynamics, or QCD) this kind of method is only possible at energies on the order of several GeV, which are generally only achievable by particle accelerators. Thus, there has been a historic struggle to understand QCD at low energy scales.

Quantum Chromodynamics

We will proceed to the theory of quarks and gluons (QCD) that describe the strong interaction of atomic nuclei by analogy with QED. First, while there is only one charge in QED, the electrical charge, there are three "color" charges in QCD, denoted r, b and g, and their negatives, $\bar{r}$, $\bar{b}$ and $\bar{g}$. Quarks carry a single color charge. Another difference arises from the fact that the gauge group is SU(3), rather than U(1). Whereas U(1) has a single element in the basis of it's Lie algebra, SU(3) has 8, which correspond to the number of "force carriers", or *gauge bosons* in the theory.

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The Lagrangian for the theory of QCD does look almost identical to that of QED, save for threenew sets of indices that we are summing over

$$\mathcal{L}_{QCD} = \sum_q \bar{\psi}_{q,i}\left((i\gamma^\mu D_\mu)^{ij} - m_q\delta^{ij}\right)\psi_{q,j} - \frac{1}{4}F^a_{\mu\nu}F_a^{\mu\nu} \qquad (5.31)$$

where the flavor index q = u, d, s, c, b, t runs over all flavors of quarks, i, j = r, b, g run over all quark color charges and $a = r\bar{b}, g\bar{r}\ldots$ runs over 8 color-anticolor pairs of gluons. The reason we sum over flavors in QCD but not QED is because quarks undergo flavor mixing [4], or they spontaneously change from one flavor to another, whereas we do not see mixing between the charged leptons' flavors e, μ, and τ[8]. The gluon field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu \qquad (5.32)$$

where f^{abc} is the structure constants of SU(3). By comparison with SU(2), the "Pauli matrices"for SU(3) are called the Gell-Mann matrices:

$$\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad \lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \qquad (5.33)$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix} \quad \frac{1}{\ } \begin{pmatrix} 1 & 0 & 0 \\ \ldots \end{pmatrix}$$

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$\lambda_6 = \square 0 \quad 0 \quad 1 \square, \quad \lambda_7 = \square 0 \quad 0 \quad -i \square, \quad \lambda_8 = \sqrt{3} \square 0 \quad 1 \quad 0 \square$

$$- gA^a_\mu \frac{\lambda_a}{2} . \qquad (5.35)$$

Now that we have enumerated what every term in the Lagrangian is, we still do not know whatthe diﬀ erence between QED and QCD, besides the fact that the latter has more indices.

Conclusion

First, quarks and gluons are never seen in isolation. They come primarily in pairs or tripletsof quarks called pions and nucleons (like protons and neutrons) that exchange gluons with one another. Additionally, gluons interact with one another, whereas photons do not. This means we can see “glue-balls”, or bound states of multiple gluons [10]. In any case, we never see net color charged particles of any kind. This phenomenon is called color confinement, and though we do not have a rigorous mathematical proof that QCD should have this property, it has become abundantly clear from experimental evidence as well as numerical evidence that this is the case [9].

A rigorous mathematical framework for much of quantum Yang-Mills theory is still very much lacking. Attempts to do so have, however, led to new and interesting insights into pure mathematics,particularly the study of three and four- manifolds [9]. Even on the practical side, the low energy dynamics in QCD are only accessible by numerical Monte Carlo calculations which require millions of hours of compute time. Indeed, many more advances in the underlying theory of physical simulation need to be better developed in order to make a broader range of phenomena in nuclear physics (among others) accessible by this method [12].

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