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A study on the Graceful Labeling Problems in Graph Theory

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Abstract:

A. Rosa proposed a new graph labelling method called ß labeling in 1966, in which the vertices are labelled with different integers ranging from 0 to m, where m is the number of edges, and each edge is labelled with the absolute difference of the labels of its end vertices, making it unique in the graph. S. W. Golomb changed ß labelling to elegant labelling, as it is known now, a few years later.

Keywords: Graceful Labelling of P_3 and $K_{1,3}$, Labelling tree, special graph.

Introduction:

A graceful labeling of a graph G is a vertex labeling $f: V \to [0, m]$ such that f is injective and the edge labeling $f_\gamma: E \to [1, m]$ defined by $f_\gamma(uv) = |f(u) - f(v)|$ is also injective. If a graph G admits a graceful labeling, we say G is a graceful graph.

Although it has been studied for 50 years, not many general results are known about graceful labeling. Most of the results are about asserting the gracefulness of a graph class since it suffices to show a graceful labeling for each graph in the class. On the other hand, results on non-gracefulness of a graph rely basically on a necessary condition only valid for Eulerian graphs or on trying to label the graph gracefully until reaching a contradiction, which is not very effective in most of the cases.

Figure 1.1: Graceful labeling of P_3 and $K_{1,3}$.

To gain some intuition on how to label a graph gracefully, let us show how to label a path graph. So, take a path graph P_n and let $V(P_n) = \{u_0, u_1, \dots, u_{n-1}\}$ be the set of vertices such that $u_{k-1}u_k \in E(P_n)$ for $0 < k < n$. Since P_n has $m = n - 1$ edges, we must label the vertices with numbers from 0 to $n - 1$ so that every number in $[1, n - 1]$ appears as an edge label. We start with edge label $n - 1$ since there is only one way to get an absolute difference equal to $n - 1$, which is having a vertex with label 0 adjacent to a vertex with label $n - 1$. Thus, let us try labeling u_0 with 0 and u_1 with $n - 1$. Next, let us try to get an edge label with value $n - 2$. There are only two possible ways to get $n - 2$ as an absolute difference: $n - 2 = |(n - 2) - 0| = |(n - 1) - 1|$. Since u_0 has no more unlabeled adjacent vertices, we can only get the edge label $n - 2$ by labeling u_2 with 1 . Going on with this strategy, our resulting labeling will be as follows:

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$$f(u_k) = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ n - \frac{k+1}{2} & \text{if } k \text{ is odd} \end{cases}$$

Now, to show that f is indeed a graceful labeling of P_n, it suffices to show that the edge label 1 appears, which is expected to appear on the last edge $u_{n-2}u_{n-1}$. If n is even, then $f(u_{n-1}) = \frac{n}{2}$ and $f(u_{n-2}) = \frac{n-2}{2}$. Hence, $f_\gamma(u_{m-1}u_{n-2}) = \frac{n}{2} - \frac{n-2}{2} = 1$. If n is odd, an andlogous argument establishes the edge label 1 . Therefore, the following proposition holds.

Proposition 1.1. The path graph P_n is graceful for all $n \geq 1$.

For a second example, we try to find a graceful labeling for the complete graph K_n. Since K_1 and K_2 are also path graphs, they are graceful. For K_3 and K_4, Figure 2.2 presents a graceful labeling for ench one.

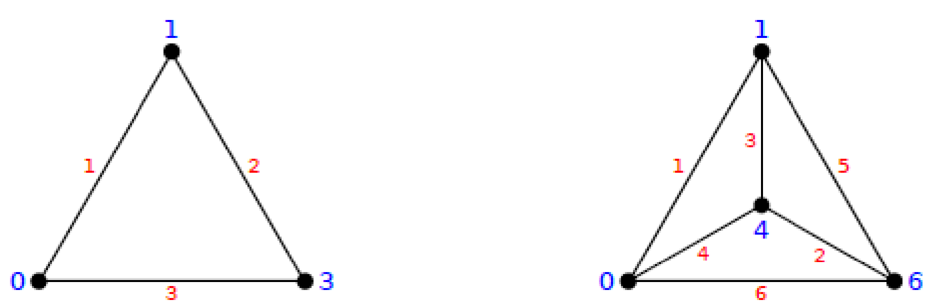

Figure 1.2: Graceful labeling of K_3 and K_4.

Before analyzing the general chase, let us first introduce a property of graceful labeling. Given a graph with a graceful labeling, if we swap every vertex label k with $m - k$, the resulting labeling is also graceful since the edge labels will not have changed: the

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end vertices of an edge with labels a and b become $m-a$ and $m-b$, and $|a-b| = |(m-a)-(m-b)|$. This is called the complementarity property.

Now, for K_n with $n > 4$, as before, we must have a vertex with latel 0 adjacent to a vertex labeled m to get the edge label m. But, in this case, every vertex is adjacent to every other vertex. Thus, we can label any vertex with 0 and any other one with m without loss of generality. To get the edge label $m-1$, we have two options $m-1 = |(m-1)-0| = |m-1|$. However, the complementarity property allows us to choose either one without loss of generality. Choosing to label a vertex with 1 , we get edge labels 1 and $m-1$. Now we need to get the edge label $m-2 = |(m-2)-0| = |(m-1)-1| = |m-2|$. We can not label a vertex with $m-1$ or 2 because it would create a duplicate edge label. Hence, our only option is to label a vertex with $m-2$, obtaining edge labels $2, m-3$ and $m-2$.

Since $m-3$ has already appeared on an edge, the next edge label we must obtain is $m-4 = |(m-4)-0| = |(m-3)-1| = |(m-2)-2| = |(m-1)-3| = |m-4|$. Again, we only have one option without creating duplicate edge labels, which is to label a vertex with 4 , obtaining edge labels $3, 4, m-6$ and $m-4$. At this point, we have labeled five vertices. However, for K_5, we would have $m-6 = 4$ as a duplicate edge label. For $n \geq 6$, the next edge label to get is $m-5$. But, all the five possible ways to get $m-5$ lead to a duplicate edge label. Therefore, there is no way to get label $m-5$ on an edge and the following proposition holds.

Proposition 1.2. The complete graph K_{n} is graceful if, and only if, $n \leq 4$. Given the initial intuition on how to gracefully label a graph, Section 2.1 presents some general results on graceful graphs and Section 2.2 shows the gracefulness of some graph

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classes.

1.2.1 General results

We start by showing a couple of results conoeming necessary conditions to the existence of a graceful labeling of a graph. The first one is a straightforward condition given by Golomb [12].

Proposition 1.3. If $G = (V, E)$ is graceful, then there exists a partition $\mathcal{P} = (A, B)$ of V such that the number of edges with onc end in A and the other in B is $\left\lceil \frac{m}{2} \right\rceil$.

Proof. Let $G = (V, E)$ be a graph with a graceful labeling f and consider the partition $\mathcal{P} = (A, B)$ of V such that $A = \{u \in V : f(u) \equiv 0(\bmod 2)\}$. Since there are $\left\lceil \frac{m}{2} \right\rceil$ odd values between 1 and m, and an odd difference is only possible by subtracting an even value from an odd one, the number of edges connecting two vertices with different parities must be exactly $\left\lceil \frac{m}{2} \right\rceil$.

Although Proposition 2.3 gives a necessary condition to the existence of a graceful labeling for a graph, it has no practical use since it would be necessary to check all the 2^{n-1} possible partitions of V to decide if a graph can axdmit a graceful labeling. A more useful necessary condition was given by Rosa [21], but it only applies to Eulerian graphs. It is known as the parity condition.

Theorem 1.4. Let G be an Eulerian graph. If $m \equiv 1,2$ (mod 4), then G is not graceful.

Proof. Suppose $G = (V, E)$ is a graceful Eulerian graph. Let $f: V \to [0, m]$ be a graceful labeling of G and $C = (u_0, u_1, \ldots, u_{m-1}, u_m = u_0)$ be an Eulerian cycle of G. Taking the sum of the edge labels of C modulo 2 , we have:

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$$\sum_{i=1}^{m} f_{\gamma}(u_{i-1}u_i) \quad = \sum_{i=1}^{m} |f(u_{i-1}) - f(u_i)|$$
$$\equiv \sum_{i=1}^{m} f(u_{l-1}) - f(u_i) \equiv 0 \ (\text{mod}2)$$

And, since C is an Eulerian cycle, i.e., the cycle C goes through each edge exactly once, and f is a graceful labeling of G, we have:

$$\sum_{c\in E} f_{\gamma}(c) = \sum_{k=1}^{m} k = \frac{m(m+1)}{2} \overset{(21)}{\equiv} 0 \ (\text{mod}2)$$

Thus, we must have $m \equiv 0{,}3(\text{mod}4)$ in order to satisfy equation (2.2). The parity condition, unlike Proposition 2.3, provides a simple way to test if an Eulerian graph can be graceful or not. And an interesting question arises: is there a graph class for which the parity condition is also a sufficient condition? As we will see, the parity condition does characterize at least one graph class.

In graph theory, it is natural to think of substructures that make a graph not satisfy a certain property, in this case being graceful. Such substructures can be subgraphs, induced subgraphs, or others, and they are called forbidden substructures for the graph class. Thus, one might think of finding forbidden substructures for the class of graceful graphs. However, Arumugam and Bagga [3] proved that every graph is an induced subgraph of a graceful graph.

Conclusion:

The graceful labeling of graphs has been a topic of research for 50 years and it still has many properties to be found. Although its primary interest was the graceful labeling of trees in order to solve Ringel's conjecture, graceful labeling of graphs gained over the years its own beauty and interest. the problem is presented, as well as the gracefulness of some

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rather simple graph classes like cycles and wheels. We also show necessary conditions to the existence of a graceful labeling for a graph, and two methods of constructing graceful graphs. In particular, one of them shows that any graph is an induced subgraph of some graceful graph

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