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APPLICATIONS OF CALCULUS OF VARIATION

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Abstract: In this paper we study about the applications of Calculus of Variations in different field.

Introduction:

The calculus of variation is an important branch of mathematics and physics. It is a powerful technique for the solution of problems in dynamics of rigid bodies, optimization of orbits and vibration problems. The applications of the calculus of variation are concerned chiefly with the determination of maxima and minima of function of a single or more variables. Its object is to find stationery values or ex/return of the given function.

The problem of calculus of variation was first solved by Jacob Bernoulli in 1698, but a general method of solving such problems was given by Euler.

Variation of a function

Let y be a real variable dependent on an independent real variable x and let f be a function of x, y and $y^1 = \frac{dy}{dx}$.

i.e., $f = f\left(x_1 y_1 y^1\right) = f\left(x_1 y(x), y^1(x)\right)$ …..(1)

Let $y + h\square(x)$ and $y^1 + h\square^1(x)) = f(x_1y_1y^1) + \left(h\alpha\frac{\partial}{\partial y} + h\alpha^1\frac{\partial}{\partial y^1}\right)f + \frac{1}{2!}\left(h\alpha\frac{\partial}{\partial y} + h\alpha^1\frac{\partial}{\partial y^1}\right)f + ...$

(Using Taylor's Theorem)

Since h is small one can neglect second and higher degree terms of h.

$$\Rightarrow f\left(x_1 y + h\alpha(x), y^1 + h\alpha^1(x)\right) - f\left(x_1 y_1 y^1\right) = h\alpha\frac{\partial f}{\partial y} + h\alpha^1\frac{\partial f}{\partial y^1}$$

Choose, $f\left(x_1 y + h\alpha(x), y^1 + h\alpha^1(x)\right) - f\left(x_1 y_1 y^1\right) = \delta f$,

$$\Rightarrow \delta f = h\alpha\frac{\partial f}{\partial y} + h\alpha^1\frac{\partial f}{\partial y^1} \qquad .....(2)$$

This is called the variation of f.

Now on taking $f = y$, we have from (2),

$$\delta y = h\alpha\frac{\partial y}{\partial y} + h\alpha^1\frac{\partial y}{\partial y^1}$$

$$= h\alpha.1 + h\alpha^1.0$$

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$\Rightarrow \delta y = h\alpha$ or $h\alpha = \delta y$ .....(3)

Again on taking $f = y^1$, we have from 2,

$$\delta y^1 = h\alpha \frac{\partial y^1}{\partial y} + h\alpha^1 \frac{\partial y^1}{\partial y^1}$$

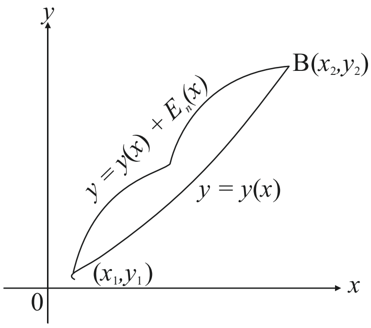
$$= h\alpha.0 + h\alpha.1 \left(\because y^1 \text{ is independent of } y\right)$$

$\delta y^1 = h\alpha^1$ or $h\alpha^1 = \delta y^1$ .....(4)

Now on using (3) and (4) in (2) we have

$$\delta f = \frac{\partial f}{\partial y}\delta y + \frac{\partial f}{\partial y^1}\delta y^1$$

.....(5)



This is the expression for variation of *f*.

The operator is δ called variational operator.

Functionals

Consider the problem of finding a curve through two points (x_1, y_1) and (x_2, y_2) whose length is a minimum (Fig. 1.1). It is same as determining the curve $y = y(x)$ for which $y(x_1) = y_1$, $y(x_2) = y_2$ such that $\int_{x_1}^{x_2} \sqrt{\left(1+y^1\right)^2}\, dx$ is a minimum.

In general terms, we wish to find the curve $y = y(x)$ where $y(x_1) = y_1$ and $y(x_2) = y_2$ such that for a given function $f(x_1 y_1 y^1)$,

$I = \int_{x_1}^{x_1} f\left(x_1 y_1 y^1\right) dx$ is a stationary value or an extremum. .....(1)

An integral such as (1) which assumes a definite value for functions of the type $y = y(x)$ is called a functional.

External of a Functional

Let us consider a functional

$$\mathrm{I} = \int_{x_1}^{x_2} f\left(x_1 y_1 y^1\right) dx \quad .....(1)$$

Associated with a function $f(x_1, y_1, y^1)$ over the interval $[x_1, x_2]$. Suppose we give a small increment h□ (x) to y. So that the function f changes to

$$f\left(x, y+\alpha(x), y^1 + h\alpha^1(x)\right) = f\left(x_1 y_1 y^1\right) \quad .....(2)$$

Where we choose, $y(x) = y + h\alpha(x)$ and

$$y^1 = y^1 + h\alpha^1(x) \quad .....(3)$$

Then the functional associated with $f(x_1, y_1, y^1)$ over the interval $[x_1, x_2]$ is given by

$$\mathrm{I}(h) = \int_{x_1}^{x_2} f\left(x_1 y_1 y^1\right) dx \quad .....(4)$$

Clearly, this functional depends on h.

Further, since I is a functional of h, the necessary condition for I to be extremum (i.e., maximum or minimum) at $h = 0$ is

$$\frac{dI}{dh} = 0 \text{ at } h = 0 \qquad \text{.....(5)}$$

From (3), we note that when $h = 0$, Y coincides with y.

□ (5) is a necessary condition that I(h) is as extremum for Y = y.

Then the curve $y = y(x)$ is called an external curve (or simply external) and the function $y = y(x)$ is called stationary function for the functional I. Now, by differentiating under integral sign, we obtain from (4),

$$\frac{d\mathrm{I}}{dh} = \int_{x_1}^{x_2} \frac{\partial}{\partial h}\left[f\left(x_1 y_1 y^1\right)\right]dx$$

On using chain rule for partial differentiation, we have

$$\frac{d\mathrm{I}}{dh} = \int_{x_1}^{x_2}\left\{\frac{\partial f}{\partial x}\frac{\partial x}{\partial h} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial h} + \frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial h}\right\}dx$$

$$= \int_{x_1}^{x_2}\left\{0 + \frac{\partial f}{\partial y}\alpha + \frac{\partial f}{\partial y^1}\alpha^1\right\}dx \ \left(\because h \text{ is ind. of } x \text{ from } (3)\right)$$

For $h = 0$, this becomes

$$\left(\frac{d\mathrm{I}}{dh}\right)_{h=0} = \int_{x_1}^{x_2}\left\{\frac{\partial f}{\partial y}\alpha + \frac{\partial f}{\partial y^1}\alpha^1\right\}dx$$

Then the condition (5) takes the form,

$$\int_{x_1}^{x_2}\left\{\frac{\partial f}{\partial y}\alpha + \frac{\partial f}{\partial y^1}\alpha^1\right\}dx = 0 \qquad \text{.....(6)}$$

This is the necessary condition for the curve $y = y(x)$ to be an external curve for the functional I. But from the previous section, we know that,

$$\delta\mathrm{I} = \delta\left[\int_{x_1}^{x_2} f^2\left(x_1 y_1 y^1\right)dx\right] = \int_{x_1}^{x_2}\delta f dx$$

$$= \int_{x_1}^{x_2}\left\{\frac{\delta f}{\delta x}\delta y + \frac{\delta f}{\delta y}\delta y^1\right\}dx$$

$$= \int_{x_1}^{x_2}\left\{\frac{\delta f}{\delta x}h\alpha + \frac{\delta f}{\delta y^1}h\alpha^1\right\}dx \text{ (by equation (3) and (4) of sec 1.2)}$$

i.e., $\delta\mathrm{I} = h\int_{x_1}^{x_2}\left\{\frac{\partial t}{\partial y}\alpha + \frac{\delta f}{\delta y^1}\alpha^1\right\}dx$

Then, the condition (6) is equivalent to the condition

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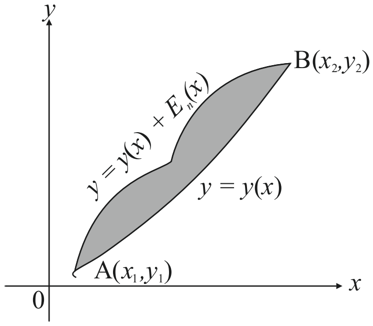
□I = 0

Hence, a necessary condition for $y = y(x)$ to be an external curve for the functional I is that □I = 0.



Variational Problem

The problem of determining an external curve $y = y(x)$ for a functional I is called a variational problem.

Let, $\mathrm{I} = \int_{x_1}^{x_2} f\left(x_1 y_1 y^1\right) dx$

be a functional associated with the function $f(x_1 y_1 y^1)$ over the interval $[x_1, x_2]$

Where $y(x_1) = y_1$ and $y(x_2) = y_2$

To find the external curve$y = y(x)$for the functional I, which passes through the two given points $[x_1, y_1]$ and $[x_2, y_2]$, we have two conditions:

(i) □I = 0 i.e., $\delta \int_{x_1}^{x_2} f\left(x_1 y_1 y^1\right) dx = 0$

(ii) $y(x_1) = y_1$, $y(x_2) = y_2$

The condition (i), which is of the form of an equation, known as variational equation. The condition (ii) which are the values corresponding to two particular values x_1 and x_2 of x, known as end conditions (or boundary conditions)].

The variational equation (i) together with the end conditions (ii) constitutes the variational problem.

Euler's equation

A necessary condition for $\mathrm{I} = \int_{x_1}^{x_2} f\left(x_1 y_1 y^1\right) dx$ ……. (I) to be as extremum is that

$$\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial y^1}\right) = 0$$

This is called Euler's equation.

Proof: Let $y = y(x)$ be the curve joining points A($x_1 y_1$) and B($x_2 y_2$) which makes I as extremum. Let

$y = y(x)$ + □$y(x)$ …………(1)

be a neighbouring curve joining these points so that at

A,$y(x_1)$ = 0 and at B, A,$y(x_2)$ = 0 …………(2)

The value of I along (I) is

$$\mathrm{I} = \int_{x_1}^{x_2} f\left[x_1 y(x) + \Sigma y(x); y^1(x) + \Sigma y^1(x)\right] dx$$

This being a function of E, is a maximum or minimum for E = 0; when

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$$\frac{dI}{dE}=0 \text{ at } E=0 \qquad \ldots\ldots\ldots\ldots(3)$$

Therefore, differentiating I under the integral sign by Leibnitz's rule, we have

$$\frac{dI}{dE}=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial x}\frac{\partial x}{\partial E}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial E}+\frac{\partial f}{\partial y^1}\frac{\partial y^1}{\partial E}\right]dx \qquad \ldots\ldots\ldots\ldots(4)$$

But E being independent of x,

$$\frac{\partial x}{\partial E}=0. \text{ Also from (1), } \frac{\partial y}{\partial E}=y(x) \text{ and } \frac{\partial y^1}{\partial E}=y^1(x)$$

Substituting these values in (4), we get

$$\frac{dI}{dE}=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}y(x)+\frac{\partial f}{\partial y}y^1(x)\right]dx$$

Integrating the second term on the right by parts, we have

$$\frac{dI}{dE}=\int_{x_1}^{x_2}\frac{\partial f}{\partial y}y(x)+\left[\left|\frac{\partial f}{\partial y^1}y(x)\right|_{x_1}^{x_2}-\int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial y^1}\right)y(x)dx\right]$$

$$=\int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y^1}\right)\right]y(x)dx$$

Since this has to be zero by (3)

$$\therefore \frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y^1}\right)=0 \qquad \ldots\ldots\ldots\ldots(\text{II})$$

Which is the desired Euler's equation.

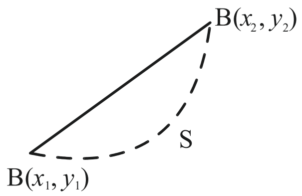

Fig. 1

Prove that the shortest distance between two points in a plane is a straight line.

Solution: Let A(x_1,y_2) and B(x_2,y_2) be the given points and S the arc length of a curve connecting than (Fig. (1)). Then

$$S=\int_{x_1}^{x_L}dS=\int_{x_1}^{x_L}\sqrt{\left(1+y^{1^2}\right)}dx$$

Now S will be minimum if it satisfies Euler's equation

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y^1}\right)=0$$

Here $f=\sqrt{1+y^{1^2}}$ which is independent of y.

i.e., $\frac{\partial f}{\partial y}=0$

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$$\therefore \frac{d}{dx}\left\{\frac{\partial}{\partial y^1}\sqrt{\left(1+y^2\right)}\right\}=0$$

or $\therefore \frac{d}{dx}\left\{\frac{y^1}{\left(1+y^{1^2}\right)}\right\}=0$

□ On integration, we have $\frac{y^1}{\sqrt{1+y^{1^2}}}$ = constant

y^1= constant, M say

Integrating, we get $y = Mx$ + C, which is a straight line, the constants M and C are determined from the fact that the straight line passes through A and B.

Find the function *y* for which the integral

$$\int_{x_1}^{x_2}\left(x^2y^{1^2}+6y^2+2xy\right)dx$$

Solution: Here, $f\left(x_1y_1y^1\right)=x^2y^{1^2}+6y^2+2xy$ ....(1)

The Euler's equation is

$$\frac{\partial f}{\partial y}-\frac{d}{dx}\left(\frac{\partial f}{\partial y}\right)=0$$

$$\Rightarrow \left(12y+2x\right)-\frac{d}{dx}\left(2x^2y^1\right)=0$$

$$\Rightarrow \left(12y+2x\right)-2x^2\frac{dy^1}{dx}-y^1+x=0$$

$$\Rightarrow -2\left\{x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-6y-x\right\}=0$$

$$\Rightarrow x^2\frac{d^2y}{dx^2}+2x\frac{dy}{dx}-6y=x$$

This is a non-homogeneous ordinary differential equation with variable co-efficients.

Put $x=e^t \Rightarrow t=\log x$

So that, $x\frac{dy}{dx}=\mathrm{D}y$ and

$x^2\frac{d^2y}{dx^2}=\mathrm{D}\left(\mathrm{D}-1\right)y$, where $\mathrm{D}=\frac{d}{dt}$

Then (2) $\Rightarrow \mathrm{D}\left(\mathrm{D}-1\right)y+2\mathrm{D}y-6y=e^t$

$$\Rightarrow \left(\mathrm{D}^2-\mathrm{D}+2\mathrm{D}-6\right)y=e^t$$

$$\Rightarrow \left(\mathrm{D}^2+\mathrm{D}-6\right)y=e^t$$

This is a non-homogenous ordinary differential equation with constant co-efficients.

$\therefore$The solution is y = C.F + P.I

To find C.F: The A.E is

$M^2 + M - 6 = 0$

$\Rightarrow (M - 2) + (M + 3) = 0$

$\Rightarrow M = 2, -3$

$\therefore$ C.F $= C_1 e^{2t} + C_2 e^{-3t}$

To find P.I: $\text{P.I} = \dfrac{1}{D^2 + D - 6} e^t = \dfrac{e^t}{1^2 + 1 - 6} = -\dfrac{e^t}{4}$

Then (S) $\Rightarrow y = C_1 e^{2t} + C_2 e^{-3t} - \dfrac{1}{4} e^t$

$$\Rightarrow y = C_1 (e^t)^2 + C_2 (e^t)^{-3} - \frac{1}{3} e^t$$

$$\Rightarrow y = C_1 x^2 + C_2 x^{-3} - \frac{1}{3} x$$

$$\Rightarrow y = C_1 x^2 + \frac{C_2}{x^3} - \frac{x}{3}$$

This is the required extremal of given integral.

Applications of Calculus of variation in Physics and mathematics:

Analytical and celestial mechanics: Formulating Newton's laws of motion, defining Lagrangian and Hamiltonian mechanics, and calculating particle paths and orbits in space.

Optimal control: Determining the best control strategy for a system, such as the path a rocket should take to minimize fuel consumption.

Differential geometry: Calculating geodesics (the shortest path between two points on a surface) and finding minimal surfaces.

Quantum mechanics: Using the variational method to approximate the ground state energy of a system.

Engineering

Structural engineering: Designing beams to be as strong as possible by determining the shape that maximizes strength.

Fluid dynamics: Analyzing fluid motion and optimizing systems like wind turbines to maximize power output.

Chemical engineering: Designing piping systems to minimize pressure drop, which saves energy and reduces costs.

Control systems: Automating systems like the exposure of a camera or the speed of a motor.

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Computer science and data science

Image processing: Using methods like total variation denoising to remove noise from images.

Machine learning: Applying variational Bayesian methods for approximating difficult integrals in statistical models.

Finite element method: Finding numerical solutions to complex differential equations for problems with irregular geometries.

Other fields

Economics: Finding the production level that maximizes profit or minimizes cost.

Medicine: Developing optimization algorithms for biological and biomedical applications.

Manufacturing: Optimizing designs to minimize the materials needed to produce an item