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Abstract

One important parametric family among the life distributions is the exponential family distributions, which play a central role within the class of all life distributions. Because of their remarkable properties, exponential distributions arise naturally in theoretical settings. It is not surprising, then, that exponential distributions have been overused in applications; but that does not diminish their importance. The importance of exponential distribution is partly due to the fact that several of the most commonly used families of life distributions are parametric extensions of this distribution. Such a parametric extension of a particular family of distributions will helps to capture the skewness and peakedness inherent in the data sets, which enables a more realistic modeling of data arising many different real life situations. Also, exponential distribution, with their constant hazard rates, form a baseline for evaluating other parametric families of distributions. One can see much more about this distribution in Balakrishnan and Basu (1995), Johnson et al. (1994), Mann et al. (1974) and Nelson and Wayne (2004). For characterizations of the exponential distribution, see

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

Galambos and Kotz (1978) and Azlarov and Volodin (1986).

The double exponential distribution (Laplace distribution), which is actually, sym- metric extension of exponential distribution to real line is a competitive model with the normal distribution. The heavy tail and the over peakedness of Laplace distri- bution than normal found applications in modeling data from various contexts such as finance, engineering, astrophysics, geographical information systems, grain size distribution, stock returns and exchange rate changes, business firm growth, humanheredity, information theory, pattern recognition, image and signal processing etc , see Howard and Vitter (1992), Lau and Post (1992), Nakayama et al. (1993), Rachev and Sengupta (1993), Alliney and Ruzinsky (1994), Wu and Fitzgerald (1995), Theodos- siou (1998), Walker and Jackson (2000), Kozubowski and Podgorski (2001), Linden (2001), Nelson (2002), Bottazzi and Secchi (2003a, b, c), Etzel et al. (2003), Gross and Levine (2003), Binia (2005), Linden (2005), Xi et al. (2005) and Sharma et al.

Introduction

In this chapter, we discuss some of the recent extensions exponential distribution on real line (generalizations of Laplace distribution) and related time series models.The Laplace distribution is considered as the one among the important statistical distributions due to its appropriateness in modeling data arising from the variety of real life situations, see Kotz et. al (2001). The density and the characteristic functions

of a Laplace random variable X are respectively,

$$f(x) = \frac{1}{2\sigma} e^{-\frac{|x|}{\sigma}}, \sigma > 0, -\infty < x < \infty, \tag{6.1.1}$$

$$\Psi_X(t) = \frac{1}{1 + t^2\sigma^2}. \tag{6.1.2}$$

The Laplace distribution is a symmetric distribution. Recently, it can be seen that the researchers are more interested in the skewed forms of symmetric distributions may be due to the fact that most of the real datasets are not symmetric. Different forms of skewed Laplace distributions can be seen in the literature. Some important

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

skewed forms of Laplace distributions are

- Asymmetric Laplace distributions obtained by the method of inverse scale factors (Skew Laplace type-1 distributions denoted as SL_1).

- Asymmetric Laplace distribution obtained by the method of hidden truncation (Skew Laplace type-2 distribution denoted as SL_2.)

- Asymmetric Laplace distribution obtained as the convolution of exponential and Laplace random variable (Skew Laplace type -3 distribution denoted as SL_3.)

Kozubowski and Podgorski(2000) introduced an asymmetric Laplace distribution by the method of inverse scale factors. The characteristic function of asymmetric Laplace distribution with skewness parameter κ is,

Note that when $\mu = 0$ that is $\kappa = 1$, corresponds to the characteristic function of symmetric Laplace distribution. Such an extension increase the fields of applications of Laplace distribution, see, Kozubowski and Podgorski (2000) and Julia and Vives-Rego (2005).

Many authors introduced non Gaussian stationary autoregressive processes and continous time Levy processes connected with the Laplace distribution, and pointed out general schemes leading to such models, which show promise in stochastic modeling. Time series models with marginal as Laplace, and $\alpha-$ Laplace distributions can be seen in Jayakumar et al. (1995) and Seetha Lakshmi et al. (2003). Jayakumar

and Kuttikrishnan (2007) introduced a time series model with asymmetric Laplace distribution (that is, skew Laplace type-1 distribution), having characteristic function (6.1.3), as marginal distribution.

Although the theory and applications of skew Laplace distributions is well developed and there is considerable literature in recent years, their application in time series modeling is not well developed. In this chapter autoregressive processes SL_2 and SL_3 distribution as marginals are developed. In Section 2, we give an overview on SL_2 distribution. First order autoregressive model with SL_2 distribution as marginals is introduced in Section 3. Skew Laplace type-3 distribution is discussed in Section 4 and related time series models are discussed in Section 5. The estimation of the pa-

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022
Research Paper

rameters involved in the process is also discussed. Section 6 is about generalizations of the SL_3 distribution and the corresponding AR(1) processes.

Skew Laplace type 2 (SL_2) distribution

Another asymmetric Laplace distribution is obtained by using Azzalini (1985)'s methodof introducing skewness into a symmetric distribution, known as method of hidden truncation, see Arnold and Beaver (2000a). A skewed Laplace probability density, by

the method of Azzallini (1985), takes the form

$$f(x) = \frac{1}{\sigma}\begin{cases} e^{-|\frac{x}{\sigma}|} - \frac{1}{2}e^{-(1+\lambda)|\frac{x}{\sigma}|} & \text{for } x \geq 0 \\ \frac{1}{2}e^{-(1+\lambda)|\frac{x}{\sigma}|} & \text{for } x < 0. \end{cases} \tag{6.2.1}$$

where $\sigma > 0$ is the scale parameter and $\lambda \geq 0$ known as skewness parameter sinceit controls skewness. Let us denote the distribution with density function (6.2.1) as SL_2 (Skew Laplace type 2) distribution. Note that $\lambda = 0$ corresponds to the parent symmetric Laplace distribution. The characteristic function of this distribution is given by

$$\Psi(t) = \frac{t + (1+\lambda^2)i}{(t+i)(t^2 + (1+\lambda)^2)} \tag{6.2.2}$$

Kozubowski and Nolan (2008) has shown that this distribution with characteristic function (6.2.2) is self decomposable whenever λ satisfies the condition $0 \leq \lambda \leq \frac{3-\sqrt{5}}{5-1}$.

Other important basic measures of this distribution are given below

$$E(X^k) = \sigma^k\Gamma(k+1)\begin{cases} 1 & \text{if k is even} \\ 1 - \frac{1}{(1+\lambda)^{k+1}} & \text{if k is odd} \end{cases} \tag{6.2.3}$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

In particular,

$$E(X) = \sigma \quad 1 \quad \frac{1}{} -$$

$$(6.2.4)(1+\lambda)^2$$

(6.2.5)

$$V(X) = \frac{\sigma^2(\lambda^4 + 3\lambda^3 + 8\lambda^2 + 8\lambda + 2)}{(1+\lambda)^4}$$

$$Skewness(X) = \frac{2\lambda(\lambda^5 + 6\lambda^4 + 15\lambda^3 + 20\lambda^2 + 15\lambda + 6)}{(\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 2)^{\frac{3}{2}}} \quad (6.2.6)$$

(6.2.7)

$$Kurtosis(X) = \frac{3(3\lambda^8 + 24\lambda^7 + 88\lambda^6 + 192\lambda^5 + 272\lambda^4 + 176\lambda^3 + 64\lambda^2 + 8)}{(\lambda^4 + 4\lambda^3 + 8\lambda^2 + 8\lambda + 2)^2}$$

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

First order autoregressive process with SL_2 as marginal distribution.

Consider the first order autoregressive process as defined in (5.2.1) where the sequence $\{X_n\}$ is SL_2 distributed with characteristic function (6.2.2) and $\{\epsilon_n\}$ is independentand identically distributed observations.

Using (5.2.1), and in terms of characteristic function we can write the characteristic function of the innovation sequence as

$$\Psi_\epsilon(t) = \frac{t+(1+\lambda^2)i}{(t+i)(t^2+(1+\lambda)^2)} \frac{(\rho t+i)(\rho^2t^2+(1+\lambda)^2)}{\rho t+(1+\lambda^2)i} \tag{6.3.1}$$

$$= \rho^2 + (1-\rho^2)\Psi_{E_{mix}}(t) \tag{6.3.2}$$

where $\Psi_{Emix}(t)$ is the characteristic function of mixture of exponential randomvari-ables and is given by

$$\Psi_{Emix}(t) = h_1\frac{1}{1+t/\eta_1} + g_1\frac{1}{1-t/\lambda_1} + g_2\frac{1}{1-t/\lambda_2} + g_3\frac{1}{1-t/\lambda_3} \tag{6.3.3}$$

where $h_1 = \frac{\rho+\sqrt{p}}{2(1+\rho\sqrt{p})}$, $\frac{1}{1+\lambda 2}$, $\eta_1 = \frac{1}{\sqrt{p}}$ and $g_1 = \frac{1-\rho^2 p}{(1+\rho)(1-\rho p)}$, $g_2 = \frac{\rho-\sqrt{p}}{2(1-\rho\sqrt{p})}$, $g_3 = \frac{-\rho^2(1-p)^2}{(1+\rho)(1-\rho p)(1-\rho^2\rho)}$, $p = $ 2

Therefore innovation sequence ϵ_n is given by

$$\epsilon_n = \begin{cases} 0 & \text{with probability} \quad \rho^2 \\ E_{mix} & \text{with probability} \quad 1-\rho^2 \end{cases} \tag{6.3.4}$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

As shown in Kozubowski and Podgorski (2008) the density function corresponding to the characteristic function (6.3.3) is given by

$$g(x) = h_1\eta_1 e^{\eta_1 x} I_{(-\infty,0)}(x) + \sum_{n=1} g_i\lambda_i e^{-\lambda_i x} I_{[0,\infty)}(x) \tag{6.3.5}$$

Skew Laplace type 3 (SL_3) distribution

Another form of skew Laplace distribution can be obtained by the convolution of symmetric Laplace and exponential distributions. This distribution is known as Skew Laplace type 3 (SL_3) distribution, see Kozubowski and Podgorski (2008). The probability density function of the SL_3 distribution is given by

$$f(x) = \begin{cases} \frac{1}{2(\sqrt{1-c^2}-c)} e^{-\frac{1}{\sqrt{1-c^2}}\left|\frac{x}{\sigma}\right|} - \frac{c}{1-2c^2} e^{-\frac{c}{\sigma}\frac{1}{\sigma}\left|\frac{x}{\sigma}\right|} & \text{for } x \geq 0 \\ \frac{1}{2(\sqrt{1-c^2}+c)} e^{-\frac{1}{\sqrt{1-c^2}}\left|\frac{x}{\sigma}\right|} & \text{for } x < 0 \end{cases} \tag{6.4.1}$$

A random variable X following Skew Laplace type 3 distribution has characteristicfunction is given by,

$$\Psi_X(t) = \frac{1}{[1+(1-c^2)\sigma^2t^2][1-ic\sigma t]}, \tag{6.4.2}$$

$\sigma > 0$, $c \in [-1, 1]$. It is denoted by $X \stackrel{d}{=} SL_3(c, \sigma)$. Whenever the parameter c=0, we obtain the standard symmetric Laplace distribution. This distribution arise as the distribution of the random variable X_λ, where,

$$X_\lambda = \frac{1}{\sqrt{1+\lambda^2}}X + \frac{\lambda}{\sqrt{1+\lambda^2}}|Y|,$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

and by denoting $c = \frac{\lambda}{1+\lambda^2}$

$\in [-1, 1]$ where X and Y are independent and identically distributed standard Laplace random variables, see Kozubowski and Podgorski (2008). The above characteristic function is actually is the characteristic function of the convolution of a Laplace random variable and an independent exponential random variable, see Jose et al (2010). That is, it is the characteristic function of the

random variable Z= L+E, where $L \overset{d}{\sim} L((1-c^2)^{\frac{1}{2}}\sigma)$. and $E \overset{d}{\sim} Exp(c\sigma)$. From (6.4.2),

it is clear that the SL_3 distribution is infinitely divisible. Next we introduce an AR(1)time series model with skew Laplace distribution as marginals.

First order autoregressive process with SL_3 as marginal distribution

Consider the AR(1) process,

$$X_n = \rho X_{n-1} + \epsilon_n, 0 < \rho < 1. \tag{6.5.1}$$

In terms of characteristic function, we obtain,

$$\Psi_\epsilon(t) = \frac{\Psi_X(t)}{\Psi_X(\rho t)} \tag{6.5.2}$$

The first order SL_3 autoregressive process is given by (6.5.1) and ϵ_n is a sequence of independent and identically distributed random variables such that X_n is stationary Markovian with SL_3 marginal distribution. Suppose that $X_n \sim SL_3(c, \sigma)$. Then

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

$$\Psi_{\epsilon}(t) = \frac{[1+(1-c^2)\sigma^2\rho^2t^2]}{[1+(1-c^2)\sigma^2t^2]}\frac{[1-ic\rho\sigma t]}{[1-ic\sigma t]} \tag{6.5.3}$$

$$= \rho^3 + \rho^2(1-\rho)\frac{1}{1-ic\sigma t} + \frac{(1-\rho^2)\rho}{2}\frac{1}{1+i\sqrt{(1-c^2)}\sigma t} + \frac{(1-\rho^2)\rho}{2}\frac{1}{1-i\sqrt{(1-c^2)}\sigma t} + (1-\rho)^2(1+\rho)\frac{1}{[1+(1-c^2)\sigma^2t^2][1-ic\sigma t]} \tag{6.5.4}$$

Therefore we can represent the innovation sequence as

$$\epsilon_n = \begin{cases} 0 & \text{with probability } \rho^3 \\ \sqrt{1-c^2}\sigma E_{2n} & \text{with probability } \frac{(1-\rho^2)\rho}{2} \\ c\sigma E_{1n} & \text{with probability } \rho^2(1-\rho) \\ -\sqrt{1-c^2}\sigma E_{3n} & \text{with probability } \frac{(1-\rho^2)\rho}{2} \\ SL_3 & \text{with probability } (1-\rho)^2(1+\rho) \end{cases} \tag{6.5.5}$$

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

where E_{in}, i=1, 2, 3 are independent and identically distributed exponential random variables.

Using (6.5.3) we can also be written as,

$$\Psi_{\epsilon}(t) = \rho^2 + \frac{(1-\rho^2)}{(1+(1-c^2)\sigma^2 t^2)}\,\rho + \frac{(1-\rho)}{(1-ic\sigma t)}. \qquad (6.5.6)$$

This implies that the innovation sequence is a convolution of a Laplace tailed random variable and an independently distributed tailed exponential random variable of Littlejohn (1994). That is, ϵ_n can be written as

$$\epsilon_n \overset{d}{=} Y_1 + Y_2 \qquad (6.5.7)$$

where Y_1 is a tailed Laplace random variable and Y_2 is a tailed exponential random variable, ie, $Y_1 \sim LT(\rho^2, (1-c^2)^{\frac{1}{2}}\sigma)$ and $Y_2 \sim ET(\rho, c\sigma)$.

Theorem 6.5.1. *The AR(1) process as defined in (6.5.1) is strictly stationary Marko-vian with SL_3 marginal distribution if and only if $\{\epsilon_n\}$'s are independent and identically distributed as defined in (6.5.5) (or (6.5.7)), provided $X_0 \overset{d}{\sim} SL_3(c, \sigma)$.*

e-ISSN 2320 –7876 www.ijfans.org
Research Paper Vol.11, Iss.9, Dec 2022

Proof: The equation (6.5.1), when it expressed in terms of characteristic function becomes,

$$\Psi_{X_n}(t) = \Psi_{X_{n-1}}(\rho t)\Psi_{\epsilon_n}(t) \tag{6.5.8}$$

on assuming stationarity and if $X_n \overset{d}{\sim} SL_3(c,\sigma)$, we obtain, $\Psi_{\epsilon}(t)$ same as (6.5.3) and

so $\{\epsilon_n\}$'s are independent and identically distributed as defined in (6.5.7).

The converse can be proved by the method of mathematical induction. From (6.5.8) and assuming $X_0 \overset{d}{\sim} SL_3(c,\sigma)$, we obtain $X_1 \overset{d}{\sim} SL_3(c,\sigma)$. Now assuming $X_{n-1} \overset{d}{\sim} SL_3(c,\sigma)$, we obtain the required result.

Another representation of the innovation random variable is obtained using the result that,

$$\frac{[1+(1-c^2)\sigma^2\rho^2t^2]\,[1-ic\rho\sigma t]}{[1+(1-c^2)\sigma^2t^2]\,[1-ic\sigma t]} = \frac{p_1}{(1-i\sqrt{1-c^2}\sigma t)} + \frac{p_2}{(1+i\sqrt{1-c^2}\sigma t)} \tag{6.5.9}$$

$$+\frac{p_3}{(1-ic\sigma t)} \tag{6.5.10}$$

where $0 < p_i < 1$, i=1,2,3 and $\sum_{i=1}^{3} p_i = 1$ is given by

$$p_1 = \frac{(1-\rho^2)\sqrt{1-c^2}}{\sqrt{\ldots}\,(1-\ldots c^2\ \ c)} + \ldots$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

$$\frac{(\sqrt{1-c^2}+c)(\rho c+(\sqrt{1-c^2}+c))}{(\sqrt{1-c^2}-c)} \qquad (6.5.11)$$

$$p = \frac{\rho c + (\sqrt{1-c^2} + c)p_1}{(\sqrt{1-c^2}-c)} \qquad (6.5.12)$$

$$p_3 = 1 - p_1 - p_2 \qquad (6.5.13)$$

Therefore we can represent the error variable ϵ_n as

$$\epsilon_n = \begin{cases} E_1 & \text{with probability } p_1 \\ -E_2 & \text{with probability } p_2 \\ E_3 & \text{with probability } p_3 \end{cases} \qquad (6.5.14)$$

where E_i's, i=1, 2 are exponentially distributed with parameter $(\sqrt{1-c^2})\sigma$ and E_3 follows exponential distribution with parameter cσ.

The joint characteristic function of (X_n, X_{n-1}), can be written as

$$\Psi_{X_n,X_{n-1}}(t_1, t_2) = E[\exp(it_1X_n + it_2X_{n-1})] \qquad (6.5.15)$$

$$= \Psi_{\epsilon}(t_2)\Psi_X(t_1 + \rho t_2) \qquad (6.5.16)$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

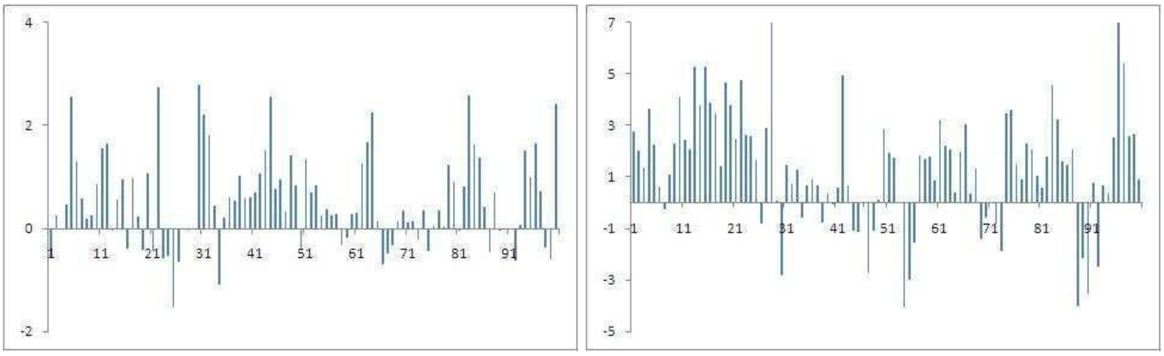

Figure 6.1: sample path of the process (6.5.1) for the parameters c=.25, $\sigma = 1$ and c=.5, σ=1.

In the case where $X_n \sim SL_3(c, \sigma)$, the above becomes

$$\Psi_{X_n,X_{n-1}}(t_1, t_2) = \frac{[1+(1-c^2)\sigma^2\rho^2 t_2^2]\,[1-ic\rho\sigma t_2][1-ic\sigma t_2]}{[1+(1-c^2)\sigma^2 t_2^2]\,[1+(1-c^2)\sigma^2\rho^2(t_1+\rho t_2)^2][1-ic\sigma(t_1+\rho t_2)]} \quad (6.5.17)$$

The joint distribution is obtain by inverting the joint characteristic function. Note that the characteristic function (6.5.17) is not symmetric in the arguments t_1 and t_2. So the process is not time reversible.

Using the AR(1) structure $X_n = \rho X_{n-1} + \epsilon_n$, we can write,

$$\Psi_{X_n}(t) = \Psi_X(\rho^n t)\prod_{k=0}^{n-1}\Psi_\epsilon(\rho^k t) \quad (6.5.18)$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

Suppose X_n $^d k=0$

$SL_3(c, \sigma)$. It can be seen that,12 2 n 2 n

$$\prod \Psi(\rho^k t) = \frac{[1 + (1 - c\ )\sigma\ \rho\ t\ ]\,[1 - ic\rho\ \sigma t]}{}$$

(6.5.19)

When $n \longrightarrow \infty, \epsilon$

$k=0$ $1 + (1 - c^2)\sigma^2 t^2]$

$[1 - ic\rho\sigma t]$

$$\Psi X_n(t) \longrightarrow \frac{}{[1 + (1 - c^2)\sigma^2 t^2][1 - ic\sigma t]}.$$

Hence X_n is asymptotically distributed as $SL_3(c, \sigma)$.

We have,

$$\Psi' X_n(t) = \frac{([1 + (1 - c^2)\sigma^2 t^2]\,([1 - ic\sigma t]ic\sigma) + [1 - ic\sigma t]2\sigma^2 t(1 - c^2)])}{([1 + (1 - c^2)\sigma^2 t^2][1 - ic\sigma t])^2} \quad (6.5.20)$$

When t=0, we obtain $E(X)=c\sigma$, $E(\epsilon_n) = (1 - \rho)(c\sigma)$.

Therefore, $E(X_n|X_{n-1} = x) = \rho x + (1 - \rho)(c\sigma)$.

Now let us look at the sample path behavior of the process discussed above. Using (6.5.14) we obtain,

$$P(X_n > X_{n-1}) = p_1 P(E_1 > (1 - \rho)X_{n-1}) + p_2 P(-E_2 > (1 - \rho)X_{n-1}) + p_3 P(E_3 > (1 - \rho)X_{n-1}). \quad (6.5.21)$$

where p_i's and E_i's are as given above. Using simple algebraic calculations, it can beshown that

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

$$P(E_1 > (1-\rho)X_{n-1}) = \sqrt{1-c^2(1-\rho)}\left[\frac{1}{2(2-\rho)[\sqrt{1-c^2}-c]} - \frac{c^2}{(1-2c^2)[\sqrt{1-c^2(1-\rho)}+c]}\right] \qquad (6.5.22)$$

and

$$P(-E_2 > (1-\rho)X_{n-1}) = \frac{\sqrt{1-c^2}}{2(2-\rho)[\sqrt{1-c^2}+c]}, \qquad (6.5.23)$$

(6.5.24)

$$P(E_3 > (1-\rho)X_{n-1}) = \frac{\sqrt{1-c^2}}{2[\sqrt{1-c^2}-c]}\left[1-\frac{c}{(1-\rho)\sqrt{1-c^2}+c}-\frac{c^2}{1-2c^2}\right]. \qquad (6.5.25)$$

On substituting (6.5.22), (6.5.23) and (6.5.25) in (6.5.21) we obtain the required probability.

Estimation of parameters can be done as follows. The parameter $'\rho'$ can be estimated from the sample auto correlation, ie. we obtain $\hat{a} = \sqrt{Corr(X_n, X_{n-1})}$. The other parameters are obtained by equating the the sample cumulants and corresponding population cumulants. The estimators are

$$\hat{\sigma} = \frac{\kappa_1}{(1-\hat{\rho})\hat{c}}$$

$$\hat{c}^2 = \frac{2\kappa_1}{\kappa_1 + \kappa_2}$$

Consider another process of the structure

$$X_n = \begin{cases} X_{n-1} & \text{with probability } p \\ \rho X_{n-1} + \epsilon_n & \text{with probability } 1-p \end{cases} \tag{6.5.26}$$

Using characteristic function we obtain the characteristic function of the innovation as (6.5.3), therefore the innovation sequence ϵ_n is distributed as in (6.5.5).

Next we discuss the higher order AR process with SL_3 as marginal distribution.The k^{th} order autoregressive process with SL_3 as marginal distribution is given by

$$\rho_1 X_{n-1} + \epsilon_n \quad \text{with probability} \quad p_1$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

$$X_n = \begin{cases} \rho_2 X_{n-2} + \epsilon_n & \text{with probability } p_2 \\ \cdot & \cdot \\ \cdot & \\ \cdot & \\ \rho_k X_{n-k} + \epsilon_n & \text{with probability } p_k \end{cases} \quad (6.5.27)$$

where $0 < p_i < 1$, i=1,2,..,k. and $\sum_{i=1}^{k} p_i = 1$ and $\{X_n, n \geq 1\}$ are SL_3 distributed. If all the $\rho_i' s$ are equal, say $\rho_i = \rho$ for i=1,2,...,k, then by using characteristic function

of SL_3 distribution, from (6.5.27), we obtain

$$\Psi_{X_n}(t) = p_1 \Psi_{X_n}(\rho t)\Psi_{\epsilon_n}(t) + p_2 \Psi_{X_{n-1}}(\rho t)\Psi_{\epsilon_n}(t) + \ldots + p_k \Psi_{X_{n-k}}(\rho t)\Psi_{\epsilon_n}(t) \quad (6.5.28)$$

Assuming stationarity we get

$$\Psi_{\epsilon}(t) = \frac{\Psi_X(t)}{\Psi_X(\rho t)} \quad (6.5.29)$$

Therefore the innovation distribution corresponding to the k^{th} order process (6.5.27)is distributed as (6.5.14).

First order autoregressive process with Generalized SkewLaplace type 3 as marginal distribution

Mathai (1993) introduced the class of generalized Laplace distribution (GL), with characteristic function

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

$$\Psi(t) = \left(\frac{1}{1+\sigma^2 t^2}\right)^{\tau}, \sigma \geq 0, \tau \geq 0 \quad (6.6.1)$$

The applications of generalized Laplace distributions in different contexts such as the production of a chemical called meltatonin in human body, solar nutrino fluxes in cosmos, growth decay mechanism like formation of sand dunes in nature etc. were discussed in Mathai (2000). The applications of generalized Laplace distribution in the field of time series modeling is discussed in Seetha Lekshmi et al. (2003) and they developed first order auto regressive process with generalized Laplace distribution as the marginal distribution. In this section, we introduce the generalized skew Laplace type 3 distribution.

A random variable X is said to follow generalized skew Laplace type 3 distribution

if its characteristic function is given by,

$$\Psi(t) = \left(\frac{1}{[1+(1-c^2)\sigma^2 t^2][1-ic\sigma t]}\right)^{\tau}, \sigma \geq 0, \tau \geq 0 \quad (6.6.2)$$

and it is denoted by $X \overset{d}{\sim}$ GSL$_3(\tau, c, \sigma)$.

From the form of the characteristic function (6.6.2) we can see that GSL$_3(\tau, c, \sigma)$ is the $\tau-$ fold convolutions of independent and identically distributed as SL$_3$ random variables. Another representation is obtained by noting that characteristic function (6.6.2) is the convolution of a generalized Laplace and a independently distributed gamma random variable. ie, a GSL$_3(\tau, c, \sigma)$ distributed random variable Z has the representation $Z = X + G$, where X is a $GL(\tau, (1-c^2)^{\frac{1}{2}} \sigma)$ and G is Gamma$(\tau, c\sigma)$ distributed random variable. When τ=n, a positive integer, then the GSL$_3(\tau, c, \sigma)$ is

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

self-decomposable being n-fold convolution of skew Laplace type 3 distribution.

An AR(1) process of the form (6.5.1), with generalized skew Laplace marginal distribution of type 3 can be construct in the same method discussed in the Section 1. The distribution of the innovation random variable ϵ_n can be represent as the distribution of the $\tau-$ fold convolution of the ζ_n where

$$\zeta_n = \begin{cases} E_1 & \text{with probability} \quad p_1 \\ -E_2 & \text{with probability} \quad p_2 \\ E_3 & \text{with probability} \quad p_3 \end{cases} \tag{6.6.3}$$

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

where $E_i's$, i=1, 2 are exponentially distributed with parameter $(\sqrt{1-c^2})\sigma$ and E_3 follows exponential distribution with parameter $c\sigma$ and p_i, i=1,2,3 is as defined in the section 2.

Conclusion

So formed distributions have important applications in the theory of time series analysis. The outline of theses is as follows

In the focus on exponentiated exponential distribution. The importance of this distribution in various real life situations and in distribution theory is discussed in Gupta and Kundu (1999). But our focus is mainly on constructing time series models for data distributed according to exponentiated exponential distri- bution. We introduce Marshal-Olkin Generalized Exponential Distribution (MOGE) and discuss many of its important properties. As an illustration, we successfully fitted the MOGE distribution for two datasets. As a generalization to the exponentiated exponential distribution we study expo- nentiated Weibull distribution in Chapter 3. Many lifetime data are of bathtub shape or upside-down bathtub shape failure rates and so the exponentiated Weibull distri- bution as a failure model is more realistic than that of distributions with monotone failure rates and plays an important role to represent such data. But, much studieshave not done in the case of exponentiated Weibull distribution. In Chapter 3 we introduce an exponentiated Weibull process and studied many important properties of this process. A discriminate study is done in between gamma distribution and exponentiated Weibull distribution and illustrated it by using two datasets. a general time series model is introduced. A strictly monotone function $\varphi(x)$, $\varphi(0) = 0$ and $\varphi(\infty) = \infty$ is used for constructing the stationary auto regressive time series models. Many of the existing time series models can be derivedas the particular case. Also we can use time series models introduced in Chapter 4 forconstructing auto regressive process for distributions having a closed form expressionfor its distribution function.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

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