SIGNAL NUMBER OF SOME GRAPHS

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ABSTRACT— A set S of vertices in a connected graph $G = (V, E)$ is called signal set of G if every vertex not in S lies on a signal path between vertices from S. A signal number is the minimum cardinality of all signal sets in G. In this paper, signal number of certain classes of graphs are determined and some of its general properties are obtained.

Keywords—signal distance, signal sets, signal number.

I INTRODUCTION

We consider here only the finite, simple, connected graph with vertex set V and edge set E. For any graph G the *order* is n and *size* is m. The degree $d(v)$ of a vertex v in $V(G)$ is the number of edges incident to v. For any vertex v in G, the open neighbourhood $N(v)$ is the set of all vertices adjacent to that v and $N[v] = N(v) \cup \{v\}$ is the closed neighbourhood of v. Let $\Delta = \Delta(G)$ and $\delta = \delta(G)$ denote for the maximum and minimum degree of G, respectively. If G be any graph, then the complement of G is obtained by $\overline{G}$. The girth of G is denoted by $c(G)$, which is the length of the shortest cycle in G. A vertex v is said to be extreme vertex of G if its neighbourhood $N(v)$ induces a complete subgraph of G. If G is a connected graph, then the distance denoted by $d(x, y)$ is the length of a shortest $x - y$ path in G. On the various study of distance in graphs, we refer to [1]. In continuation, Kathiresan et.al introduced a new distance parameter known as signal distance of graphs [4]. The signal distance $d_{SD}(u, v)$ between the pairs u and v is defined by $d_{SD}(u, v) = \min\left\{d(u, v) + \sum_{w \in V(G)} (degw - 2) + (degu - 1) + (degv - 1)\right\}$ where S is the path connecting u and v, $d(u, v)$ be the length of path S and in the sum $\sum_{w \in V(G)}$ runs over all the internal vertices between u and v in the path S. The $u - v$ signal path of length $d_{SD}(u, v)$ is also called geosig. A vertex v is known as lie on a geosign P if v is an internal vertex of P. In [6], author introduce the notation that $L[x, y]$ consists of x and y and all vertices lying on some $x - y$ geosig of G and for a non-empty set $S \subseteq V(G)$, $L[S] = \cup_{x,y \in S} L[x, y]$. A set $S \subseteq V(G)$ is said to be a signal set of G if $L[S] = V(G)$. The minimum cardinality of a signal set is known as signal number and is denoted by $sn(G)$. [2] A set S as a subset of $V(G)$ is known as geodetic set if $I[S] = V(G)$. The minimum cardinality of a geodetic set of G is known as geodetic number and is denoted by $g(G)$. The undefined notations and symbols we refer [2,5]. A star graph is a complete bipartite graph $K_{1,n-1}$ of order n. A bistar graph $B(m, n)$ is obtained from K_2 by attaching m edges in one vertex and n-edges in the other vertex. The following theorems is very much useful for the following sections.

Theorem 1.1. [6] For extreme vertex of G belongs to every signal set of G.

Theorem 1.2. [6] $sn(G) = 2$ if and only if there exist vertices u, v such that v is an u-signal vertex of G.

II MAIN RESULT

Theorem: 2.1. Let G be any connected graph of order n. Then $2 \leq sn(G) \leq n$.

Proof: Any signal set we need at least two vertices to find the signal distance so $sn(G) \geq 2$. Also all the vertices of G together form a signal set of G. Thus $sn(G) \leq n$. Hence $2 \leq sn(G) \leq n$.

Remark: 2.2. The bounds in Theorem 2.1 are sharp. The two end vertices or the vertices of degree 1 in any path graph forms a signal set which is unique of minimum cardinality of P_n and so $sn(P_n) = 2$. Also for the complete graph K_n $(n \geq 2)$, every vertices are extreme vertices. Therefore by Theorem 1.1, $sn(K_n) = n$. Moreover, all of the inequalities in Theorem 2.1 can be strict. Consider the graph G given in Figure 2.1. Here we find out that $n = 7$. Here $S = \{v_1, v_2, v_5, v_6\}$ is a signal set of minimum cardinality. Hence $sn(G) = 4$. Thus $2 < sn(G) < n$.

Theorem 2.3. For the star graph $G = K_{1,n-1}$ ($n \geq 3$), $sn(K_{1,n-1}) = n - 1$.

Proof. Let $n \geq 3$ and let $V(K_{1,n-1}) = \{v_1, v_2, \ldots, v_{n-1}\}$, where v is the only vertex of degree $n-1$ and each pendant vertex v_i $(1 \leq i \leq n-1)$ is adjacent to v. Let $S = \{v_1, v_2, \ldots, v_{n-1}\}$. Then by Theorem 1.1, $sn(K_{1,n-1}) \geq |S| = n - 1$. Since v lies on every v_i-v_j geosig path between vertices from S, that S itself form a signal set of G and hence $sn(K_{1,n-1}) = n - 1$

Theorem 2.4. If G is a bistar graph, then $sn(G) = n - 2$.

Proof. Let $G = B_{m,n}$ be a bistar graph with $m, n \geq 1$. Take $V(G) = \{u, v, u_i, v_j; 1 \leq i \leq m, 1 \leq j \leq n\}$ and $E(G) = \{uv, uu_i, vv_j; 1 \leq i \leq m, 1 \leq j \leq n\}$. Let S be a minimum signal set of G. Then by Theorem 1.1, $\{u_1, u_2, \ldots, u_m, v_1, v_2, \ldots, v_n\} \subseteq S$. Since S itself from a signal set of G, we conclude that $sn(G) = n - 2$.

Theorem 2.5. Let K_n $(n \geq 2)$ by any complete graph and e be any edge of G. Then $sn(K_n') = 2$ where $K_n' = K_n - \{e\}$ is the edge deleted graph.

Proof. Consider an edge $e = uv$ in $E(K_n)$ and an edge deleted graph $K_n' = K_n - \{e\}$. Let $S = \{u, v\} \subseteq V(K_n')$. Then it is clear that every vertices w in $K_n' - S$, there exists a $u - v$ geosig path of length 2 contains that w. Therefore, that S itself from a signal set of K_n' and so $sn(K_n') \leq |S| = 2$. By Theorem 2.1, we conclude that $sn(K_n') = 2$.

Theorem 2.6. Let $G = K_n - \{e_i, e_j\}$ is the graph obtained from K_n ($n \geq 4$), by removing edges $\{e_i, e_j\}$ for $i \neq j$. Then

$$sn(G) = \begin{cases} 2 & if\ e_i\ and\ e_j\ are\ adjacent \\ n-2 & if\ e_i\ and\ e_j\ are\ non-adjacent \end{cases}$$

Proof. Let $G = K_n - \{e_i, e_j\}$, where e_i and e_j are distinct edges of K_n.

Case(i) e_i and e_j are adjacent. Then the edge deleted graph G contains one vertex v is of degree $n-3$, two vertices (say u, w) of degree $n-2$ and the remaining vertices of degree $n-1$. Consider $S = \{u, v\}$. Then $d(u, v) = 2$ in G and any $u - v$ geosig path contain exactly one internal vertex. Also the signal distance of u and v with any disjoint $u - v$ geosig path is same and so that S is a signal set of G. Hence $sn(G) \leq 2$. By Theorem 2.1, we conclude $sn(G) = 2$.

Case(ii) e_i and e_j are non-adjacent.

Let $e_i = uv$ and $e_j = xy$ for some $u, v, x, y \in V(G)$. Consider $S = \{x, y\} = \{u, v\}$. Then the geosig path between vertices from S contains only u, v, x, y. This geosig path does not contain the vertices of degree $n-1$ in G. Consider now $S_1 = S \cup T$, where $T \subseteq V(G) - S$ having $n-4$ vertices and those vertices have degree $n-1$. By Theorem 1.1, T contain in every signal set of G because they are extreme vertices. Also $L[S_1] = V(G)$. Therefore that S_1 forms a signal set of G and so $sn(G) \leq |S_1| = n-2$. Since any two vertices in T are adjacent and two find a signal set we need at least two non-adjacent vertices and so $Sn(G) \geq n-2$. Hence $sn(G) = n-2$.

Theorem 2.7. If G is a connected graph with a cut-vertex v, then every signal set of G contains at least one vertex from each component $G - v$.

Proof. Let v be any cut vertex of G. Let $G_1, G_2, \dots, G_k$ $(k \geq 2)$ be the components $G - v$. Let S be any signal set of G. On the contrary, we assume that S contain no vertex from a component say G_i $(1 \leq i \leq k)$. Let u be a vertex on G. Then by Theorem 1.1, u is not an extreme vertex of G. Since S is a signal set of G, there exist vertices x, y in S such that u lies on a $x - y$ geosig path $P: x = u_0, u_1, u_2, \dots, u, \dots, u_l = y$ such that $u \neq x, y$. Since v is a cut vertex of G that subpath $x - u$ of P and the subpath $u - y$ of P both contains the vertex v. It sharp that P is not a path, which is a contradiction. Hence every signal set of G contains at least one vertex from every component of $G - v$.

Corollary 2.8. Let G be a connected graph with cut-vertices and let S be a signal set of G. Then every branch of G contains an element of S.

Theorem 2.9. No cut-vertex of G belong to any minimum signal set of G.

Proof. Let v be any cut vertex of G and let S be any minimum signal set of G. Then by Theorem 2.7 every component of $G - v$ contain an element of S. Suppose$v \in S$. Let X and Y be any two disjoint components of $G - v$. Let $x \in X$ and $y \in Y$. Then v is an internal vertex of an $x - y$ geosig path. Consider $S' = S - \{v\}$. Then it is clear that every internal vertex lies on an $x - v$ geosig path also lies on an $x - y$ geosig path. It follows that S' is a signal set of G with cardinality less than S, which is a contradiction. Since S is of minimum cardinality. Then $v \notin S$.

Corollary 2.10. For any tree T with k-end vertices, $sn(T) = k$.

Proof. T contains k-end vertices and remaining all are cut vertices.

3. REALISATION RESULTS

Theorem 3.1. For any three positive integer r, d and k such that $r \leq d \leq 2r$ and $a \geq 2$, there exists a connected graph G with $rad(G) = r, diam(G) = d$ and $sn(G) = a$.

Proof: For $r = 1$, we have either $d = 1$ or $d = 2$. Suppose $d = 1$, then consider $G = K_a$ and $Sn(G) = a$. Suppose $d = 2$. Then consider that $G = K_2 + (K_{a-1} \cup K_1)$. Then it is clear that G satisfies the required conditions. If $r = d = 2$, then consider $G = K_{a,a+1}$. One can observe easily that $sn(G) = a$. Now for $r \geq 3$, we divide into the following cases. **Case (i).** $r = d$. Let G be a graph obtained from disjoint union of cycle .$C_{2r}: v_1, v_2, \dots, v_{2r}, v_1$ and a complete graph K_{a-2} by joining v_1 and v_{2r} to all vertices of $K_{a-2.}$ Then we easily verified that $radG = r$ and $diamG = r$. Also $V(K_{a-2}) \cup \{v_1, v_{r+1}\}$ is a minimum signal set of G. Hence $sn(G) = a$. **Case (ii).** $r < d$. Let u_i. Consider a cycle $C_{2r}: v_1, v_2, \dots, v_{2r}, v_1$ of order $2r$ and a path $P_{d-r+1}: u_0, u_1, \dots, u_{d-r}$ of order $d - r + 1$. Let H be a graph obtained that from the graphs C_{2r} and P_{d-r+1} by identifying v_i and u_0. Also $w_1, w_2, \dots, w_{a-2}$ of $a - 2$ vertices to H and join each w_i $(1 \leq i \leq a - 2)$ to u_{d-r+1}. A new graph G is obtained and it shown in Figure 2.3. We observe that $radG = r$ and diam$G = d$. Also $S = \{w_1, w_2, \dots, w_{a-2}, u_{d-r}\}$ is the set of all extreme vertices of G and so by Theorem 1.1, $sn(G) \geq a - 1$. But there is no signal set exists with $a - 1$ vertices. Therefore $sn(G) \geq k$. Now it is clear that $S \cup \{v_{r+1}\}$ is a signal set of G and hence $sn(G) = a$.

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