e-ISSN 2320 –7876 www.ijfans.org

Research Paper

Vol.11, Iss.9, Dec 2022

A Study on One Vertex Union of Alternate Shells Other Labeling in Graph Theory.

*1 .Kiran P. S.
M.Phil. Scholar, Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai-72
*2 Dr. V. Vetrivel
Associate Professor, Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai-72.
ps_kiran@ymail.com. vetrivel.maths@bharathuniv

Address for Correspondence

*1 .Kiran P. S.
M.Phil. Scholar, Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai-72
*2 Dr. V. Vetrivel
Associate Professor, Department of Mathematics, Bharath Institute of Higher Education and Research, Chennai-72.
ps_kiran@ymail.com. vetrivel.maths@bharathuniv

Abstract:

In this research paper we prove that the one vertex union of alternate shells with a path at any common level with chords $G(2n_i, n_i - 2, k, l_c)$, is graceful and admits an α-valuation, for $k \geq 1, n_i \geq 3$ and $1 - i \leq k$. The graph one vertex union of alternte shells with a path at any common level (with or without chords) $G(2n_i, n_i - 2, k, l)$, is proved to be cordial, for $n_i = n \geq 3$, $k \geq 1$ and $1 \leq i \leq k$. As a corollary we also prove that the graph $G(2n, n - 2, k, l, 2t)$ is cordial, for $k \geq 1, n \geq 3$ and $1 \leq t \leq (k \quad 1)(n - 1)$.

Keywords: Graph labelling, shell graphs, division cordial labelling graph, shells path, vertex union. etc.

Introduction:

One Vertex Union of Alternate Shells with A Path

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

In this section we describe the one vertex union of alternate shells with a path at any common level (with or without chords) $G(2n_i, n_i - 2, k, l)$, by properly defining the common level vertices in the k altemate shells $C(2n_i, n_i - 2), 1 \leq i \leq k$.

Let $C(2n_1, n_1 - 2), C(2n_2, n_2 - 2), \ldots, C(2n_k, n_k - 2)$ be any k altemate shells, without loss of generality, we assume that $n_1 \leq n_2 \leq \cdots \leq n_k$. The alternate shell $C(2n_i, n_i - 2)$ is called i^{th} alternate shell.

Let $\cup_{i=1}^{k} C(2n_i, n_i - 2)$ denote the one vertex union of k alternate shells $C(2n_i, n_i - 2)$'s, for $1 - i \leq k$, obtained by merging the apex of each of these k alternate shells together. Let v_0 denotes the common vertex of $\overset{k}{\cup} C(2n_i, n_i - 2)$, and let $v_{i,j}$ be the vertex of the i^{th} alternate shell $C(2n_i, n_i - 2)$ at the j^{th} level, where $1 \leq j \leq 2n_i - 1$. The $(i-1)^{\text{th}}$ alternate shell $C(2n_{i-1}, n_{i-1} - 2)$ and the i^{th} alternate shell $C(2n_i, n_i - 2)$ are said to be adjacent altemate shells in $\cup C(2n_l, n_l - 2)$, for $2 \leq i \leq k$

We arrange the vertices of each alternate shell $C(2n_i, n_i - 2)$ excluding the apex v_0 in the one vertex union of altemate shells Ui $C(2n_i, n_i - 2)$ with certain hierarchy, so that k suitable common level vertices (with chords and without chords) can be identified for joining them by a path P_{2k-1}.

`Level arrangement of first and second alternate shells

` First arrange the vertices of the first alternate shell $C(2n_1, n_1 - 2)$ as a chain $v_{1,1}, v_{1,2}, \ldots, v_{1,2n_1-1}$ such that $v_{1,1}$ is in the bottom level vertex and $v_{1,2m-1}$ is in the top level vertex of $C(2n_1, n_1 - 2)$. Now we arrange the vertices of $C(2n_2, n_2 - 2)$ as a chain $v_{2,1}, v_{2,2}, \ldots v_{2,2n_2-1}$ such that the top level vertex $v_{1,2m_1-1}$ of the first alternate shell $C(2n_1, n_1 - 2)$ and the top level vertex $v_{2,2n_2-1}$ of the second alternate shell

$C(2n_2, n_2 - 2)$ are in the same level, so that the bottom level vertex $v_{1,1}$ of the first alternate shell $C(2n_1, n_1 - 2)$ and

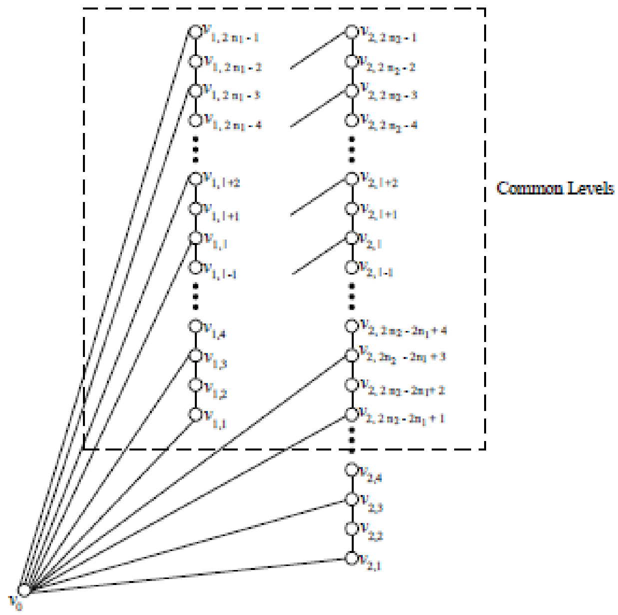


Figure 1.1 Level arrangement of $C(2n_1, n_1 - 2)$ and $C(2n_2, n_2 - 2)$

the vertex $v_{2,2n_2-2n_1-1}$ of the second alternate shell $C(2n_2, n_2 - 2)$ are in the same level. Observe that there are n_1 pairs of vertices $\langle v_{1,2n-1}, v_{2,2n_-1} \rangle$, $\langle v_{1,2n_1-3}, v_{2,2n_2-3} \rangle, \dots, \langle v_{1,1}, v_{2,2n_2-2n_1-1} \rangle$ are in the same levels with chords, and $n_1 - 1$ pairs of vertices $\langle v_{1,2n_1-2}, v_{2,2n_2-2} \rangle, \langle v_{1,2n_-4}, v_{2,2n_2-4} \rangle, \dots, \langle v_{1,2}, v_{2,2n_2-2n_3-2} \rangle$ are in the same level without chords, see Figure 4.1.

Level arrangement of i^{th} alternate shell

When $3 \leq i \leq k$ and i odd, arrange the i^{th} alternate shell $C(2n_i, n_i - 2)$ as a chain $v_{i,1}, v_{i,2}, \dots, v_{i,2n_1-1}$ such that the bottom level vertex $v_{i,1}$ of the $(i-1)^{)}$ altemate shell $C(2n_{t-1}, n_{t-1} - 2)$ and the bottom level vertex $v_{i,1}$ of the i^{th} alternate shell $C(2n_i, n_l - 2)$ are in the same level, so that the top level vertex $v_{i-1,2n_{-1}-1}$ of the $(i-1)^{\text{t}}$ alternate shell

$C(2n_{i-1}, n_{i-1} - 2)$ and the vertex $v_{i,2n_{+-1}}$ of the i^{th} alternate shell $C(2n_i, n_i - 2)$ are in the same level.

Thus, there are $n_{v,1}$ pairs of vertices $\langle v_{1-1,1}, v_{t,1}\rangle$, $\langle v_{L-1,3}, v_{i,3}\rangle, \ldots, \left\langle v_{\langle 1,2n_{-1}-1}, v_{i,2n_{j-1}-1}\right\rangle$ are in the same levels with chords, and there are $n_{t-1} - 1$ pairs of vertices $\langle v_{i,1,2}, v_{i,2}\rangle$, $\langle v_{i-1,4}, v_{i,4}\rangle, \ldots, \langle v_{i-1,2n_{-1}-2}, v_{i,2n_{-1}-2}\rangle$ are in the same levels without chords.

When $3 \le i \le k$ and i even, arrange the vertices of the i^{th} alternate shell $C(2n_i, n_i - 2)$ as a chain $v_{i,1}, v_{i,2}, \ldots, v_{i,2n,-1}$ such that the top level vertex $v_{j-1,2,-1-1}$ of the $(i-1)^{\text{th}}$ alternate shell $C\left(2n_{j-1}, n_{i-1} - 2\right)$ and the top level vertex $v_{i,2n_1-1}$ of the i^{it} alternate shell $C(2n_i, n_i - 2)$ are in the same level, so that the bottom level vertex $v_{i-1,1}$ of $(i-1)^{\text{th}}$ alternate shell $C(2n_{l-1}, n_{i-1} - 2)$ and the vertex $v_{i,2n-2n_{-1}-1}$ of the i^{th} alternate shell $C(2n_i, n_i - 2)$ are in the same level.

Thus, there are n_{h-1} pairs of vertices $\langle v_{t-1,2n-1-1}, v_{i,2n,-1}\rangle$, $\langle v_{t-1,2n_{n-1}-3}, v_{i,2n_3-3}\rangle, \ldots, \langle v_{t,1,1}, v_{i,2n,-2n_{-1}-1}\rangle$ are in the same levels with chords, and $n_{l-1} - 1$ pairs of vertices $\langle v_{i-1,2n_{2-1}-2}, v_{l,2n_1-2}\rangle$,

$\langle v_{l-1,2n_{-1}-4}, v_{t,2n_1-4}\rangle, \ldots, \langle v_{i-1,2}, v_{i,2n_{-}-2n_{-1}-2}\rangle$ in the same levels without chords.

We refer the above hierarchical level arrangement of vertices of the alternate shell $C(2n_i, n_i - 2)$ in $\cup_{i=1}^{k} C(2n_i, n_i - 2)$ is called Top-Bottom -Level arrangement (TBL-arrangement). In the TBL-arrangement of $\cup_1^k C(2n_i, n_i - 2)$ there are n_1 levels of vertices in each alternate shell which are in the common level with chords and $n_1 - 1$ levels of vertices in each alternate shell which are in the common level without chords, the vertices

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

in the common levels with chords are called common level vertices with chords, and the vertices in the common level without chords are called common level vertices without chords. Observe that, there exists n_1 different sets of common level vertices with chords where each set containing a vertex from a distinct alternate shell $C(2n_i, n_i - 2)$, for $1 \leq i \leq k$, and there exists $n_1 - 1$ different sets of common level vertices without chords where each set containing a vertex from a distinct alternate shell $C(2n_i, n_i - 2)$, for $1 \leq i \leq k$.

Let ${}_{i=1}^{k}C(2n_i, n_l - 2)$ be one vertex union of k alternate shells $C(2n_i, n_i - 2)$'s, $1 \leq i \leq k$, with TBL- arrangement, choose k vertices, each from a distinct alternate shell $C(2n_i, n_i - 2)$, such that they are in any fixed common level l (with or without chords). Join these k vertices by a path P_{2k-1} such that the i^{th} alternate shell $C(2n_i, n_i - 2)$ at the $(2i-1)^{\text{th}}$ vertex of P_{2k-1}. The vertex w_i of the path P_{2k-1} is called middle vertex between the i^{thi} alternate shell $C(2n_i, n_i - 2)$ and the $(i+1)^{\text{th}}$ alternate shell $C(2n_{i-1}, n_{i-1} - 2)$, for $1 \leq i \leq k$ 1. The graph thus obtained is $G(2n_i, n_i - 2, k, l)$.

In particular, we denote the graph $G(2n_1, n_l - 2, k, l)$ as $G(2n_i, n_i - 2, k, l_c)$ whenever the path P_{2k-1} taken only at any common level l with chords, see Figure 4.2. We observe the following two observations from the graph $G(2n_i, n_i - 2, k, l)$.

Observation 1.1.

In each of the alternate shell $C(2n_i, n_i - 2)$ excluding the apex v_0 there is a path $P_{2n_i - 1}$ of length $2n_i - 1$ in $G(2n_i, n_i - 2, k, l)$, for $1 \leq i \leq k$. In the path P_{2k-1} of $G(2n_l, n_i - 2, k, l)$ joining the k alternate shells with alternate vertices, there are k vertices which are common to the path P_{2k-1} and the k alternate shells, called shared vertices of

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

the path P_{2k-1} in $G(2n_i, n_i - 2, k, l)$, and the $(k-1)$ middle vertices w_i 's $(1 \leq i \leq k - 1)$ which lies only on the path P_{2k-1}.

Observation 1.2.

When $n_l = n$, for $1 \leq k$, then we denote the graph $G(2n_l, n_t - 2, k, l)$ as $G(2n, n-2, k, l)$. We anange the k copies of the alternate shells $C(2n, n-2)$ in $G(2n, n-2, k, l)$ as first, second, ..., k th copy. Observe that there are $2n - 2$ pairs of vertices which are in the common levels between any two adjacent copies were not joined by path P_{2k-1}. Join $2tP_3$ paths between $2t$ pairs of vertices where each pair of vertex is selected from any adjacent altemate shells in the same common level, where $1 \leq t \leq (k - 1)(n\ 1)$. The graph thus obtained is denoted by $G(2n, n-2, k, l, 2t)$.

1.3 One Vertex Union of Alternate Shells With A Path At Any Common Level With Chords Is Graceful

In this section, we show that the graph $G(2n_l, n_1 - 2, k, l_e)$ one vertex union of k altemate shells $C(2n_t, n_t - 2)$'s with a path P_{2k-1} at any common level l with chords is graceful and admits an α-valuation, for $n_t \geq 3, 1 \leq i \leq k$ and $k \geq 1$.

Theorem 1.1. For $k \geq 1, n_t \geq 3$ and $1 \leq i \leq k$, the graph $G(2n_t, n_t - 2, k, l_c)$ is graceful.

Proof. Let $G(2n_i, n_v - 2, k, l_c)$ be the one vertex union of k altemate shell $C\left(2n_t, n_f - 2\right)$'s with a path P_{2k-1} at any common level l with chords, for $n_r \geq 3, 1 \leq i \leq k$ and $k \geq 1$. Observe that

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

$$| V\left(G\left(2n_1, n_f\ 2, k, l_e\right) \right| - \sum_{t=1}^{k} (2n_l, 1) \Bigg| k = 2\sum_{i1}^{k} n_1, \text{ and}$$

$$| E\left(G(2n_1, n_i - 2, k, l_c) \right| = \sum_{i1}^{k} (3n_i - 2) + 2(k-1) = 3\left(\sum_{i1}^{k} n_i\right) - 2.$$

Let $m =| E(G(2n_1, n_x - 2, k, l_c) |$.

For the convenience of graceful labeling, we rename the vertices of $G(2n_i, n_i - 2, k, l_c)$ as shown in Figure 1.3.

Define $\phi: V\big(G(2n_1, n_t - 2, k, l_c)\big), \{0,1,2, \dots, m\}$ by

$$\phi(v_0) = 0,$$

$$\phi(v_j) = \begin{cases} m - (j-1), \text{ for } 1 \le j \le \sum_{i1}^{k} n_l, \text{ and} \end{cases}$$

$$\phi(u_j) - \begin{cases} m + 1 + j\sum_{i=1}^{\varepsilon} n_i, \text{ for } 1 \le j \le \left(\sum_{i+1}^{k} n_i\right) 1 - \end{cases}$$

From the above vertex labeling, the set $\phi(v_j)$, for $l - j - \sum_{i1}^{k} n_r\}$ form a monotonically decreasing sequence and the set $\{\phi(u_j)$, for $1 = j = \sum_{t1}^{k} n_t$ | form a monotonically increasing sequence

The following Corollary 1.1 is an immediate consequence of Rosa's theorem that if a graph G with m edges has an α-labeling, then there exists a cyclic decomposition of the edges of the complete graph K_{2am-1} into sub graphs isomorphic to G, where a is an arbitrary natural number.

Corollary 1.1. The complete graphs K_{2ax-1} can be decomposed into sub graphs isomorphic to $G(2n_t, n_t - 2, k, l_c)$, where $m = \big|E\big(G(2n_l, n_s - 2, k, l_c)\big)\big|$, and a is an arbitrary positive integer.

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

The corollary 1.2 follows from the result of El-Zanati's theorem that if G has m edges and admits an α-labeling then $K_{am,hm}$ can be partitioned into sub graphs isomorphic to G for all positive integers a and b.

Corollary 1.2. The edges of the complete graphs $K_{ax,bm}$ can be partitioned into sub graphs isomorphic to $G(2n_l, n_t - 2, k, l_c)$, where a and b are arbitrary positive integer and $m = \left|E\left(G(2n_t, n_t - 2, k, l_e)\right)\right|$.

1.4 One Vertex Union of Copies of Alternate Shells With A Path At Any Level Is Cordial

In this section, we show that the graph $G(2n, n - 2, k, l)$ the one vertex union of k copies of alternate shell $C(2n, n - 2)$ with a path $P_{\Sigma k-1}$ at any level l (with or without chords) is cordial, for $n = 3$ and $k \geq 1$. We also show that the graph $G(2n, n - 2, k, l, 2t)$ is cordial, for $n \geq 3, k \geq 1$ and $1 \leq t \leq (k \quad 1)(n \quad 1)$.

Theorem 1.2. For $n \geq 3$ and $k \geq 1$, the graph $G(2n, n - 2, k, l)$ is cordial.

Proof. Let $G(2n, n - 2, k, l)$ one vertex union of k copies of the alternate shell $C(2n, n - 2)$ with a path P_{2k-1} at any level l (with or without chords). For the convenience of cordial labeling describe the vertices of $G(2n, n - 2, k, l)$ as shown in Figure 4.2, where $n_1 = n_2 = \cdots = n_k = n$.

Observe that

$$|V(G(2n, n - 2, k, l))| = 2kn$$
$$|E(G(2n, n - 2, k, l))| = 3kn - 2$$

Define $\phi: V(G(2n, n - 2, k, l)) \rightarrow \{0,1\}$ by

$$\phi(v_0) = 0,$$

e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

Labeling of the first copy in $G(2n, n-2, k, l)$

For $1 \leq j \leq 2n-1$,

Let $\phi(v_{1,j}) = \begin{cases} 1, \text{ if } j = 1,2(\bmod 4) \\ 0, \text{ if } j = 0,3(\bmod 4) \end{cases}$

Labeling of the i^{t} copy in $G(2n, n-2, k, l)$

$$\text{Let } \phi(v_{t,j}) = 1 - \left(\phi(v_{l-1,j})\right), \text{ for } 2 \leq i \leq k, 1 \leq j \leq 2n-1.$$

Labeling of middle vertices in the path P_{2k-1}

$$\text{Let } \phi(w_i) = \begin{cases} 1, \text{ for } 1 - i - k\ 1 \text{ and } i \text{ odd} \\ 0, \text{ for } 1 \leq i \leq -1 \text{ and } i \text{ even} \end{cases}$$

Let V_0 and V_1 denotes the set of vertices of $G(2n, n-2, k, l)$ were assigned the labels 0 's and 1 's respectively.

Let E_0 and E_1 denotes the set of edges of $G(2n, n-2, k, l)$ were having the labels 0 's and l's respectively.

Let $A = \cup_1^k A_1$, where A be the set of all edge labels of the edges of the i^{th} copy of the alternate shell which are adjacent to v_0, that is, $A_1 = \{v_0 v_{i,j} - E(C(2n, n-2))$, for $1 \leq j \leq 2n-1\}$

Let B be the set of all edge labels of the edges of the paths $P'_{2n-1}5$ in each of the altemate shell $C(2n, n-2)$.

Let C be the set all edge labels of the edges of the path P_{2k-1} at any level l, where $l \in \{1,2,3, \ldots, 2n-1\}$.

It follows from the Table 4.1 that the graph $G(2n, n-2, k, l)$ is cordial.

Corollary 1.3. For $k \geq 1$ and $n \geq 3$, the graph $G(2n, n-2, k, l, 2t)$ is cordial, where $1 \leq t \leq (k \quad 1)(n \quad 1)$.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

Proof. The graph $G(2n, n-2, k, l)$ is cordial. Observe that, the labels of the corresponding vertices in the sense that if a vertex u in a copy has label 0 then its adjacent copy the vertex u has the label 1 and vice-versa in $G(2n, n-2, k, l)$. Therefore joining any two adjacent copies of an altemate shell by a path P_3 with middle vertex having the label 0 or having the label 1 will give one of the edge label as 1 and the other edge label as 0 .

Therefore, tP_3 paths having middle vertex labeled 0 's and tP_3 paths having middle vertex labeled 1 's between any adjacent alternate shells in $G(2n, n-2, k, l)$. Hence, $G(2n, n-2, k, l, 2t)$ is cordial.

1.5 Illustrative Examples

Here we give below examples to illustrate the labeling that are given in the proof of the theorems proved in this chapter.

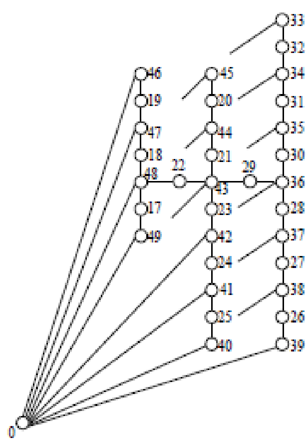
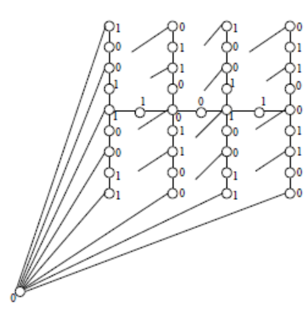


Figure 1.5 Graceful labeling of one vertex union of C(8, 2), C(12, 4), C(14, 5) with a path P5 at level 3 with chords

Figure 1.4 and Figure 1.5 are illustrative examples of the labelings given in the Proof of the Theorem 4.1.



e-ISSN 2320 –7876 www.ijfans.org
Research Paper
Vol.11, Iss.9, Dec 2022

Figure 1.6 Cordial labeling of G(10, 3, 4, 6)

Figure 1.6 and Figure 1.7 are illustrative examples of the labelings given in the Proof of the Theorem 1.2.

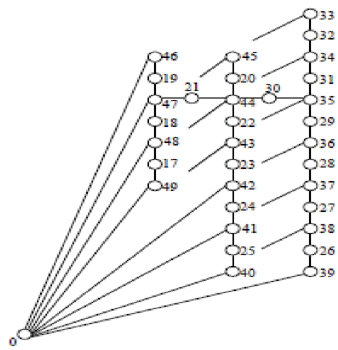


Figure 1.4 Graceful labeling of one vertex union of C(8, 2), C(12, 4), C(14, 5) with a path P_5 at level 5 with chords

Figure 1.8 is an illustrative example of the labelings given in the Proof of the Corollary 1.3.

Conclusion

In Section 1.2 of this research papter, we have shown that the graph $G(2n_i, n_i - 2, k, l_c)$ the one vertex union of k alternate shells with a path P_{2k-1} at any common level l with chords is graceful. We feel that tendency towards having the graceful labeling of $G(2n_i, n_i - 2, k, l)$ the one vertex union of k alternate shells with a path P_{2k-1} at any common level l without chords seems to be negative. It prompts to ask the following question.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

Is $G(2n_i, n_i - 2, k, l)$ the one vertex union of k alternate shells with a path P_{2k-1} at any common level l without chords graceful, for $k \geq 1, n_l \geq 3$ and $1 \leq i \leq k$?

In Section 4.3 of this chapter, we have shown that $G(2n, n - 2, k, l)$ is cordial. It appears that proving the cordialness of $G(n_i, n_i - 3, k, l)$, for arbitrary n_i 's, seem to be hard to establish. So we conclude this chapter with the following question.

2. Is $G(n, n_i - 3, k, l)$ the one vertex union of k alternate shells with a path P_{2k-1} at any common level l (with or without chord) is cordial, for all $n_i \geq 3, 1 \leq i \leq k$ and $k \geq 1$.

References

[1] Acharya B. D. and Gill M. K. (1981), 'On the index of gracefulness of a graph and the gracefulness of two-dimensional square lattice graphs', Indian J. Math., 23, pp. 81-94.

2. Aldred R. E. and McKay B. D. (1998), 'Graceful and harmonious labelings of trees', Bull. Inst. Combin. Appl., 23, pp. 69-72.

3. Andar M., Boxwala S. and Limaye N. (2002), 'Cordial labelings of some wheel related graphs, J. Combin. Math. Combin. Comput., 42, pp. 203 -208.

4. Andar M., Boxwala S. and Limaye N. (2002), 'A note on cordial labelings of multiple shells', Trends Math., pp. 77-80.

5. Ayel J. and Favaron O. (1984), 'Helms are graceful', Progress in Graph Theory (Waterloo, Ont.), Academic Press, Toronto, Ont. pp. 89-92.

6. Baca M., Miller M., MacDougall J., Slamin and Wallis W., 'On certain valuations of graphs, preprint.

7. Balakrishnan R. and Kumar R. (1994), 'Decomposition of complete graphs into isomorphic bipartite subgraphs', Graphs and Combin., 10, pp. 19-25.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

8. Balakrishnan R. and Kumar R. (1994), 'Existence and non-existence of certain labelings for the graph 2 *c* 2 *n K K* ', Utilitas Math., 46, pp. 97-102.

9. Bermond J. C.(1979), 'Graceful graphs, radio antennae and French windmills', Graph Theory and Combinatorics, Pitman, London, pp. 18-37

10. Bhat-Nayak V. N. and Gokhale S. K. (1986), 'Validity of Hebbare's conjecture', Utilitas Math., 29, pp. 49-59.