Lattice Identities On The Lattice Of Subgroups Of The Group Of 2x2 Upper Triangular Matrices Of A Matrix Group Over Finite Fields

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Abstract

In this paper our main focus is to study various lattice identities satisfied by the lattice of subgroups of the upper triangular matrices of the group of 2 x 2 matrices over Z_p under matrix multiplication modulo 'p' where p is prime and p= 2,3,5 and 7

The properties verified are modularity, super solvability, 0- Distributivity, Consistency, Pseudo 0-distributivity, super 0- distributivity, GD condition, distributivity and simple.

Keywords – Modularity, distributivity, simple, congruence.

Introduction – The main aim in this paper is to check the properties of Lattices of subgroups of the upper triangular matrices of the group of 2x2 matrices over Z_p under matrix multiplication modulo p, where p is prime and p= 2,3,5 and 7.

In 2015, D. Jebaraj Thiraviam has given the structure and checked some properties of subgroup lattices of the groups of 2x2 matrices over Z_p having determinant value 1 under matrix multiplication modulo p , where p is one of the prime numbers 2,3 5 and 7. This has motivated us to investigate the lattice of subgroups of the group of 2x2 upper triangular matrices over Z_p for which we have given the lattice structures in the paper [14]"On the lattice of subgroups of the upper triangular matrices of a matrix group over finite fields."

Preliminaries

The following definitions are used in the paper.

Definition 1.1- In the Poset (P, ≤), a covers b or b is covered by a (in notation, a $\succ$b or b $\succ$a) if and only if b<a and, for no x, b<x<a.

Definition 1.2– An element 'a' is an atom, if a $\succ$0 and a dual atom, if a $\prec$1.

Definition 1.3 – A lattice is said to be modular if whenever a ≤ c

av (b ∧ c) = (a v b) ∧ c, for all a, b, c ϵ L

Definition 1.4 – A lattice is said to be super solvable, if it contains a maximal chain called an M-chain in which every element is modular. By a modular element m in a lattice L, we mean x v (m ∧y) = (x v m) ∧ y whenever x ≤ y in L.

Definition 1.5- A lattice L is said to be 0- distributive if for all x, y, z ϵL, whenever x ∧ y=0 and x ∧ z =0 then x∧ (y v z) = 0.

Definition 1.6- An element a of a lattice is called join-irreducible if x v y=a implies x=a or y=a.

Definition 1.7 – A lattice L is said to be consistent if whenever j is a join-irreducible element in L, then for every x ϵ L, x v j is join-irreducible in the upper interval [x,1].

Definition 1.8 – A lattice L is said to be pseudo-0 distributive if for all x, y, z ϵ L, x ʌ y = 0, x ʌ z =0 imply that (x v y) ʌ z = y ʌ z.

Definition 1.9 – A lattice L is said to be super 0- distributive if for all x, y, z ϵ L, x ʌ y = 0 implies (x v y) ʌ z= (x ʌ z) v (y ʌ z)

Definition 1.10 – The lattice L with 0 satisfies the general dis-jointness property (GD) if x ʌ y = 0 and (x v y) ʌ z = 0 imply x ʌ (y v z) = 0.

Definition 1.11 – A lattice L is said to be distributive if (x v y) ʌ z = [(x ʌz) v (y ʌ z) and (x ʌ y) v z = [(x v z) ʌ (y v z)] for all x, y, z ϵ L.

Definition 1.12 – An equivalence relation ϴ on a lattice L is called a congruence relation on L if and only if (a_0, b_0) ϵ ϴ and (a_1, b_1) ϵ ϴ imply that $(a_0 \wedge a_1, b_0 \wedge b)$ ϵ ϴ and $(a_0 \vee a, b_0 \vee b)$ ϵ ϴ.

Definition 1.13 – The collection of all congruence relations on L, is denoted by Con L,and it is an algebraic lattice with respect to set inclusion relation.

Definition 1.14- If a lattice L has only two trivial congruence relations namely ω, the diagonal and τ = L x L, then L is said to be simple (eg.,M_3 is simple)

Definition 1.15 – If Con L contains a unique atom, then we say that L is sub-directly irreducible (eg., N_5 is sub-directly irreducible).

Results –

- Any modular lattice is consistent.
- Every modular lattice is super solvable.

Lattice structures of the lattices of subgroups of the upper triangular matrices of the group of 2 x 2 matrices over Z_p under matrix multiplication modulo p, where p is a prime and p=2,3,5 and 7 are displayed below.

Throughout the paper we denote the lattice of all subgroups of the group of upper triangular 2x2 matrices over Z_p by $L_u(G)$.

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
 Journal UGC CARE Listed (Group-I) Volume 12, Issue 01 Jan 2023

Fig 1

$L_u(G)$, $G=Z_2$

Fig 2

$L_u(G)$, $G=Z_3$

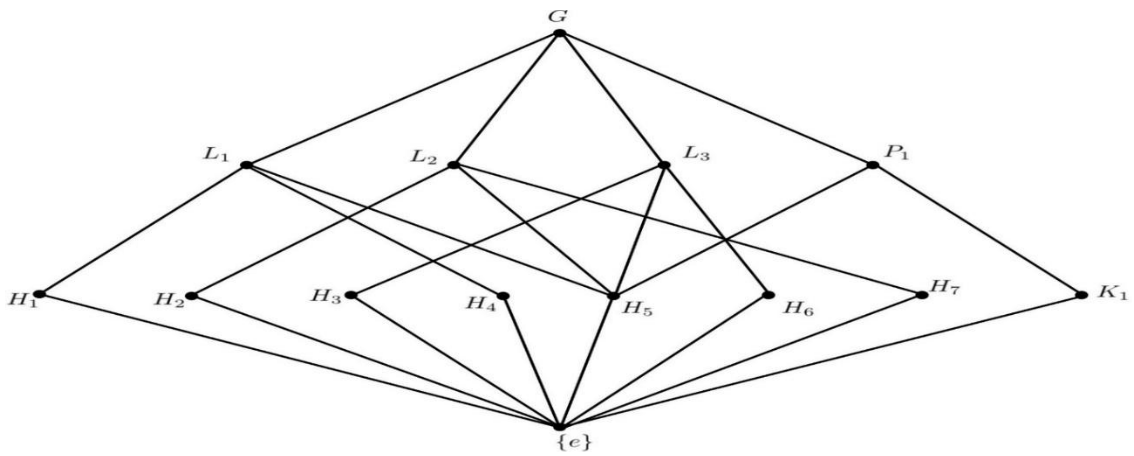


Fig.3

$L_u(G)$, $G=Z_5$

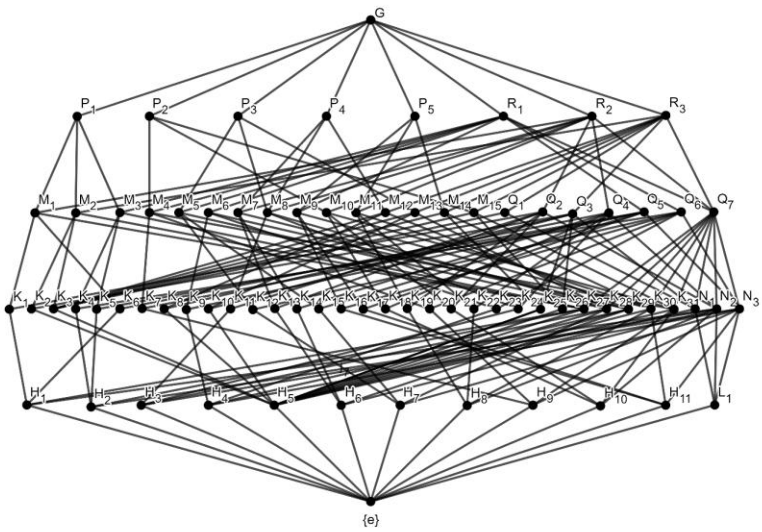
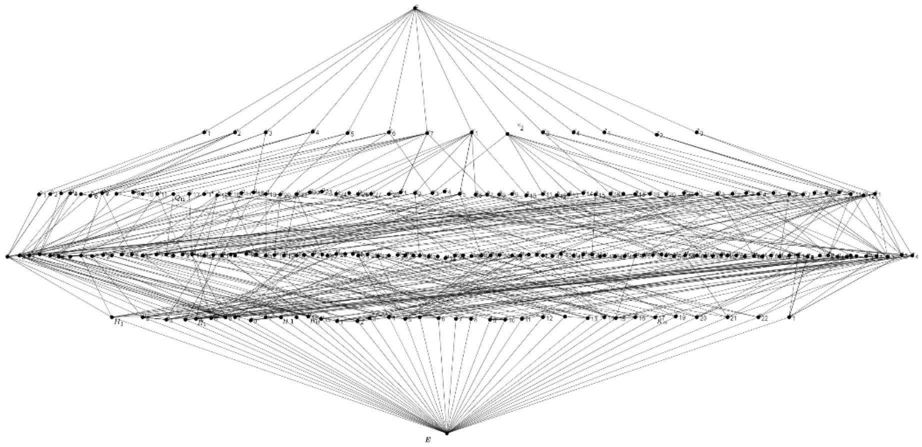


Fig 4

$L_u(G)$, G = Z_7



Lemma 1.1 – When p≤3 $L_u(G)$ is modular.

Proof – From the figure 1 and 2 we observe that whenever $x \leq z$ in $L_u(G)$

$x \vee (y \wedge z) = (x \vee y) \wedge z$ for every $y \in L_u(G)$.

Therefore, we conclude that $L_u(G)$ is modular when p≤ 3.

Lemma 1.2- When p≤3, $L_u(G)$ is consistent.

Proof – Since any modular lattice is consistent, by the previous lemma, we see that $L_u(G)$ is consistent when p≤ 3.

Lemma 1.3 – When p≤3, $L_u(G)$ is super solvable.

Proof- Since every modular lattice is super solvable, we conclude that, when p≤ 3, $L_u(G)$ is super solvable.

Lemma 1.4 – When p=3 the lattice $L_u(G)$ is not pseudo- distributive.

Proof- From fig 2 $H_1, H_3, H_4 \in L_u(G)$,

$H_3 \wedge H_4 = \{e\}$, $H_1 \wedge H_3 = \{e\}$

$(H_3 \vee H_4) \wedge H_1 = G \wedge H_1$

$= H_1$

$H_4 \wedge H_1 = \{e\}$

Therefore, $(H_3 \vee H_4) \wedge H_1 \neq H_4 \wedge H_1$.

Therefore, the lattice $L_u(G)$ is not super 0- distributive.

Lemma 1.5 – When p=3, the lattice $L_u(G)$ is not super 0- distributive.

Proof – From fig 2, $H_3, H_6, H_7 \in L_u(G)$

$$H_6 \wedge H_7 = \{e\}$$

$$(H_6 \vee H_7) \wedge H_3 = G \wedge H_3$$

$$= H_3$$

$$(H_6 \wedge H_3) \vee (H_7 \wedge H_3) = \{e\} \vee \{e\}$$

$$= \{e\}$$

Therefore $(H_6 \vee H_7) \wedge H_3 \neq (H_6 \wedge H_3) \vee (H_7 \wedge H_3)$.

Therefore, the lattice $L_u(G)$ is not super 0-distributive.

Lemma 1.6- When p=3, general dis -jointness condition is not satisfied.

Proof – From fig 2, $H_5, H_6, H_7 \in L_u(G)$

$$H_5 \wedge H_6 = \{e\} \text{ and } (H_5 \vee H_6) \wedge H_7 = L_3 \wedge H_7$$

$$= \{e\}$$

$$H_5 \wedge (H_6 \wedge H_7) = H_5 \wedge G$$

$$= H_5$$

$$\neq \{e\}.$$

Therefore, general dis-jointness condition is not satisfied in $L_u(G)$ when p=3.

Lemma 1.7 – When p=3, $L_u(G)$ is simple.

Proof- We observe that L(G) is atomistic, so if $\Theta \in [\text{Con } L(G)]$, then if $(x, y) \in \Theta$ and $x \leq y$, there exists an atom $a \in L_u(G)$ such that $a \leq y$ and a is not less than or equal to x.

Therefore $(\{e\}, a) \in \Theta$.

To prove that $L_u(G)$ is simple, it is enough to verify whether there is any proper principal congruence generated by an element of the form $(\{e\}, a)$ where a is an atom in $L_u(G)$.

Now, $\Theta (\{e\}, H_1) = \omega \cup \{\{e\}, H_1), (H_1, \{e\}), (\{e\}, L_1), (L_1, \{e\}), (H_4, L_2), (L_2, H_4), (\{e\}, L_3), (\{e\}, G)\}$

$$= L_u(G) \times L_u(G)$$

$\Theta (\{e\}, H_2) = \omega \cup \{\{e\}, H_2), (H_2, \{e\}), (H_1, L_1), (L_1, H_1), (\{e\}, L_1), (H_4, G), (H_5, G), (\{e\}, G)\}$

$$= L_u(G) \times L_u(G)$$

$\Theta (\{e\}, H_4) = \omega \cup \{\{e\}, H_4), (H_1, \{e\}), (\{e\}, L_2), (\{e\}, G)\}$

$$= L_u(G) \times L_u(G)$$

Similarly, $\Theta (\{e\}, H_5) = L_u(G) \times L_u(G)$

$\Theta(\{e\}, K_1) = \omega \cup \{\{e\}, K_1), (K_1, \{e\}), (H_7, G), (H_6, G), (\{e\}, G)\}$

$= L_u(G) \times L_u(G)$

Therefore, $L_u(G)$ has no proper congruence relation.

Therefore, $L_u(G)$ is simple.

We tabulate the subgroups of G, when p=5 in the order in which they lie in different maximal subgroups(co-atoms).

Intervals [{e}, P_i] in $L_u(G)$, i = 1,2,3,4,5

order	Subgroups
16	P_1
8	M_1, M_2, M_3
4	$K_1, K_2, K_3, K_4, K_5, K_6, K_{27}$
2	H_1, H_2, H_5
1	{e}

order	Subgroups
16	P_2
8	M_4, M_{10}, M_{15}
4	$K_3, K_7, K_{18}, K_{20}, K_{26}, K_{27}, K_{31}$
2	H_5, H_9, H_{10}
1	{e}

order	Subgroups
16	P_3
8	M_5, M_{10}, M_{13}
4	$K_3, K_{10}, K_{11}, K_{12}, K_{15}, K_{24}, K_{28}$
2	H_3, H_5, H_7
1	{e}

order	Subgroups
16	P_4
8	M_6, M_7, M_{12}
4	$K_3, K_8, K_9, K_{13}, K_{14}, K_{23}, K_{29}$
2	H_4, H_5, H_6
1	{e}

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
Research Paper Journal UGC CARE Listed (Group-I) Volume 12, Issue 01 Jan 2023

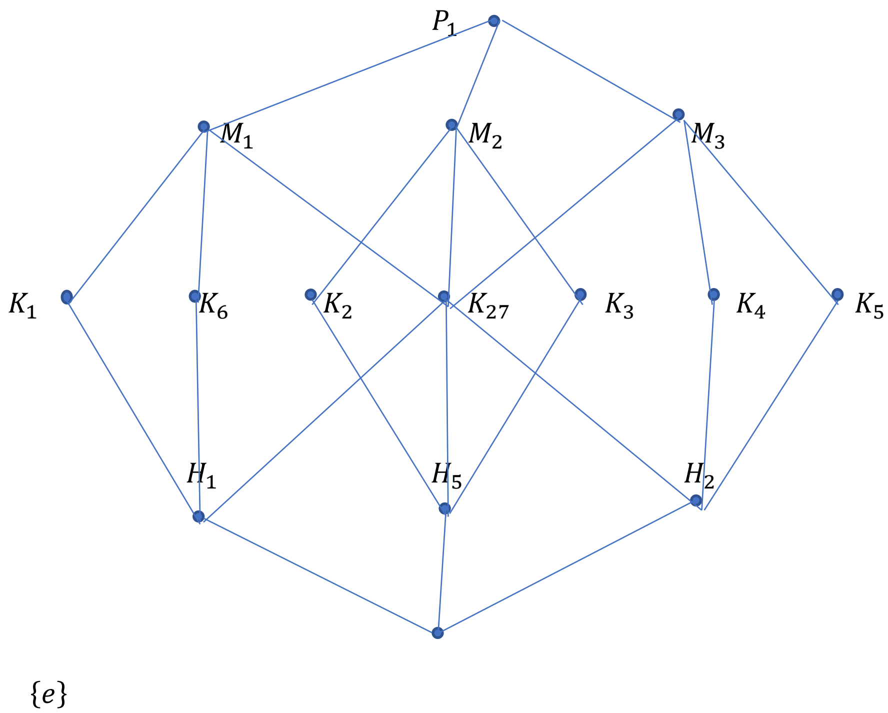
order	Subgroups
16	P_5
8	M_9, M_{11}, M_{14}
4	K_3, K_{16}, K_{17}, K_{19}, K_{22}, K_{25}, K_{30}
2	H_5, H_8, H_{11}
1	{e}

Each P_i is of order 16.

We observe that the number of subgroups of orders 2, 4, and 8 below each P_i is 3, 7 and 3 respectively and the lattice structure of the intervals [{e}, P_i] are isomorphic.

Typically, we display it for P_1 as given below.

Fig 5



<u>**Lemma 1.7**</u>- Each interval [{e}, P_i] satisfies the general dis-jointness condition, i= 1,2,3,4,5.

Proof- Since there is no such pair exist to satisfy the hypothesis of the general dis-jointness condition, obviously the GD condition is true in each interval [{e}, P_i] , i= 1,2,3,4,5

<u>**Lemma 1.8**</u> – Each interval [{e}, P_i] is super modular, i=1,2,3,4,5.

Proof – From fig 5, we observe that each interval [{e}, P_i] satisfies the identity

$(a \vee b) \wedge (a \vee c) \wedge (a \vee d) = a \vee [b \wedge c \wedge (a \vee d)] \vee [c \wedge d \wedge (a \vee b)] \vee [b \wedge d \wedge (a \vee c)]$ for all a, b, c, d $\in$ L

Therefore, the interval [{e}, P_i] is super modular i =1,2,3,4,5.

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Lemma 1.9 – Each interval $[\{e\}, P_i]$ is modular.

Proof – From fig 5, we observe that whenever $a \leq c$ in $[\{e\}, P_i]$ $a \vee (b \wedge c) = (a \vee b) \wedge c$ for all $b \epsilon [\{e\}, P_1]$

Therefore $[\{e\}, P_i]$ is modular.

In the same manner, $[\{e\}, P_i]$ is modular for every i= 1,2 ,3 4,5.

Lemma 1.10 – Each interval $[\{e\}, P_i]$ is consistent, i= 1,2,3,4,5.

Proof- Since any modular lattice is consistent, by previous lemma we see that $[\{e\}, P_1]$ is consistent. In the same manner $[\{e\}, P_i]$ is consistent for every i= 1, 2, 3, 4, 5.

Lemma1.11 – Each interval $[\{e\}, P_i]$ is not distributive for every i = 1, 2, 3,4 ,5.

Proof – From fig 5, we observe that $H_1, H_2, H_5 \epsilon [\{e\}, P_i]$

$$H_1 v(H_2 \wedge H_5) = H_1 v\{e\}$$

$$= H_1$$

$$(H_1 v H_2 \wedge (H_1 \vee H_5) = M_1 \wedge P_1$$

$$= M_1$$

Therefore, $H_1 v(H_2 \wedge H_5) \neq (H_1 \vee H_2) \wedge (H_1 \vee H_5)$.

Therefore, we conclude that the interval $[\{e\}, P_i]$ is not distributive.

Lemma 1.12 – Each interval $[\{e\}, P_i]$ is not 0- distributive for every i= 1, 2, 3, 4, 5

Proof – From figure 5, we observe that

$H_1, H_2, H_5 \epsilon [\{e\}, P_i]$

$H_1 \wedge H_2=\{e\}$ and $H_1 \wedge H_5=\{e\}$

$$H_1 \wedge (H_2 v H_5) = H_1 \wedge H_{27}$$

$$= H_1$$

$$\neq \{e\}$$

Therefore, we conclude that the interval $[\{e\}, P_i]$ is not 0- distributive for every i= 1, 2,3, 4, 5.

Lemma 1.13- Each interval $[\{e\}, P_i]$ is not Pseudo 0- distributive for every i= 1, 2, 3, 4,5.

Proof – From Fig 5, we observe that

$H_1, H_2, H_5 \epsilon [\{e\}, P_i]$

$H_1 \wedge H_2=\{e\}$ and $H_1 \wedge H_5=\{e\}$

$$(H_1 v\ H_2) \wedge H_5= K_{27} \wedge H_5$$

$$= H_5$$

$$(H_2 \wedge H_5) = \{e\}$$

$\neq H_5$

Therefore, $(H_1 \vee H_2) \wedge H_5 \neq (H_2 \wedge H_5)$.

Therefore, we conclude that the interval $[\{e\}, P_i]$ is not pseudo 0-distributive for every i= 1,2,3,4,5.

Lemma 1.14 - Each interval $[\{e\}, P_i]$ is not super 0- distributive for every i= 1, 2, 3, 4,5

Proof - From Fig 5, we observe that

$$H_1, H_2, H_5 \in [\{e\}, P_i]$$

$$H_1 \wedge H_2 = \{e\}$$

$$(H_1 \vee H_2) \wedge H_5 = K_{27} \wedge H_5$$

$$= H_5$$

$$(H_1 \wedge H_5) \vee (H_2 \wedge H_5) = \{e\} \vee \{e\}$$

$$= \{e\}$$

$$\neq H_5$$

Therefore, $(H_1 \vee H_2) \wedge H_5 \neq (H_1 \wedge H_5) \vee (H_2 \wedge H_5)$.

Therefore, we conclude that the interval $[\{e\}, P_i]$ is not super 0-distributive for every i= 1,2,3,4,5.

Lemma 1.15 – When p=7 the lattice $L_u(G)$ is not 0- distributive.

Proof - From fig 4, we observe that

$$H_1, H_2, H_3 \in L(G),$$

$$H_1 \wedge H_2 = \{e\} \text{ and } H_2 \wedge H_3 = \{e\}$$

$$H_1 \wedge (H_2 \vee H_3) = H_1$$

$$\neq \{e\}$$

Therefore, we conclude that when p=7 the lattice is not 0-distributive.

Lemma 1.16 – When p=7 the lattice $L_u(G)$ is not Pseudo-0-distributive.

Proof - From fig 4, we observe that $K_{19}, K_{20}, N_1 \in L(G)$

$$K_{19} \wedge K_{20} = \{e\} \text{ and } K_{19} \wedge N_1 = \{e\}$$

$$(K_{19} \vee K_{20}) \wedge N_1 = T_4 \wedge N_1$$

$$= N_1$$

$$\neq \{e\}$$

Therefore, the lattice $L_u(G)$ is not Pseudo 0- distributive.

Lemma 1.17 – When p=7 $L_u(G)$ is not modular.

Proof – From fig 4, we observe that S_1, U, $W_1 \in L_u(G)$

$$S_1 \vee (U \wedge W_1) \wedge (U \vee W_1) = (S_1 \vee R_2) \wedge Y_1$$

$$= G \wedge Y_1$$

$$= Y_1$$

$$S_1 \wedge (U \vee W_1) \vee (U \wedge W_1) = (S_1 \vee Y_1) \vee R_2$$

$$= H_1 \vee R_2$$

$$= R_2$$

$$\neq Y_1$$

Therefore, the lattice $L_u(G)$ is not modular.

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