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A Study on Perfect matching Bipartite Graph

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Abstract

In this paper, we introduce a corresponding between bipartite graphs with a perfect matching and digraphs, which implicates an equivalent relation between the extendibility of bipartite graphs and the strongly connectivity of digraphs. Such an equivalent relation explains the similar results on *k*-extendable bipartite graphs and *k*-strong digraphs. We also study the relation among *k*-extendable bipartite graphs, *k*-strong digraphs and combinatorial matrices. For bipartite graphs that are not 1-extendable and digraphs that are not strong, we prove that the elementary components and strong components are counterparts.

Key words: *k*-extendable, strongly *k*-connected, indecomposable, irreducible, strong component, elementary component

Introduction

One of the amazing features of Graph theory is its recognition in all fields of science and engineering. Graph theory is a fertile area of mathematical research. Domination is one of the most important concepts attracting researchers. There are many variations of domination in

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the literature such as independent domination, total domination, connected domination, paired domination just to name a few. In this series, a vertex-dominating cycle is defined as a cycle in which every vertex of the graph is adjacent to at least one vertex on the cycle. Dominating cycles find immense applications in networks.

The objective of this thesis is to introduce a new kind of dominating cycle called perfect matching dominating cycle, characterize, study the nature of these cycles and analyze some parameters associated with it in undirected, simple, connected graphs with a perfect matching. Some of the exciting new applications that are directly associated with perfect matching dominating cycles have also been explored. Perfect matching sequences, perfect matching dominating cycles and perfect matching minor for simple connected graphs G are defined. A perfect matching M in a graph G of even order, say $2n$, is a set of mutually non-adjacent edges, which covers all vertices of G. A non-repeated finite alternating sequence of edges of G from M and $E - M$ such that each edge is adjacent with the edge preceding and following it, starting and ending at the same edge of M is called perfect matching sequence (PM-sequence). The subgraph with the edges of such a PM sequence of length $2n$, denoted as GM, contains a cycle called the perfect matching dominating cycle ($pmdc$). The graph obtained by contracting all the edges of M is called a PM minor, denoted as GM .

The existence of such $pmdc$ is characterized using Hamilton cycles. The necessary and sufficient condition for the existence of a $pmdc$ is that the PM- minor is Hamiltonian. Apart from finding such cycles in general graphs, enumeration of $pmdcs$ is also carried out for some classes of graphs. It is always convenient to represent graphs using matrices for further processing. Hence the matrix of perfect matching and the reduced matrix of the perfect matching are defined. Some graph parameters related to $pmdc$ are also discussed.

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Spanning trees play a very important role in any network since it is a basic structure of the network with minimal number of edges but ensuring connectivity. Spanning trees associated with $pmdcs$ are also defined. One very desirable feature expected out of a graph in the study of combinatorial design theory is edge decomposition with certain structural property. This is taken into account and a new kind of decomposition, called n-sun decomposition is offered. The cycle in the n-sun is a $pmdc$ and hence an n-sun decomposition can be regarded as a type of $pmdc$ decomposition. The study of $pmdc$ in graph products like cartesian, strong and tensor products is done. Since strong product is the union of cartesian and tensor, cartesian and tensor products are given more attention. It is interesting to observe that not all product graphs have $pmdc$ for a given choice of perfect matching. Some classes of product graphs which are non-Hamiltonian are identified to have $pmdc$.

Two applications of perfect matching dominating cycles and its allied spanning trees are discussed. The first application finds the number of spanning trees containing a given Keknle structure in fullerene molecules. The second application is on Bluetooth devices, in which a procedure is given to find spanning tree structure with maximum simultaneous active piconets in the statement.

This thesis concludes that PM-sequences and $pmdc$ spanning trees will play a vital role in network. Also, $pmdc$ spanning trees will help in message passing from a vertex to all other vertices, with a maximum congestion of 3 links at a node and hence better than using conventional spanning trees where we cannot always expect the congestion to be less. Also there is a lot of scope for future work based on the concepts introduced and analyzed in this thesis.

There is a well-known equivalent property between the 1-extendibility of G and the strong connectivity of D.

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Theorem 1.1. *([8], Exercise 4.1.5) Let G be a bipartite graph and M a perfect matching of G. Then D* = *D*(*G*,*M*) *is strong if and only if G is 1-extendable.*

The following is another interesting relation between *G* and *D*.

Theorem 1.2. *([8], Exercise 4.3.3) Let G be a bipartite graph with a unique perfect matching M. Then D* = *D*(*G*,*M*) *is acyclic.*

In this paper we further discuss the relation between *G* and *D*, as well as their relations with combinatorial matrices.

Extendibility versus Connectivity

Below is a generalization of Theorem 1.1, which has been stated in [12] without a proof.

Theorem 2.1. *Let G be a bipartite graph and M a perfect matching of G. Then D* = *D*(*G*,*M*) *is k-strong if and only if G is k-extendable.*

We prove Theorem 2.1 in this section and show some interesting applications of it. We need Menger's Theorem in our proof.

Theorem 2.2. *(Menger [10]) Let D be a digraph. Then D is k-strong if and only if* $|V(D)| \geq k+1$ *and D contains k internally vertex disjoint (s,t)-paths for every choice of distinct vertices* $s,t \in V$.

Actually we use an equivalent form of Menger's Theorem. Further more, we only need the following weaken form, which appears as an exercise in [2].

Lemma 2.3. *([2], Exercise 7.17) Let D be a k-strong digraph. Let* $x_1, x_2, \ldots, x_{k-1}, y_1, y_2, \ldots, y_{k-1}$ *be distinct vertices of D, then there are k independent paths in D, starting at* x_i, $0 \leq i \leq k-1$ *and ending at* y_j, $0 \leq j \leq k-1$.

Now comes the proof of **Theorem 2.1.**

Proof. Let *D* be *k*-strong. We use induction on *k* to prove that *G* is *k*-extendable. When $k = 1$, the conclusion follows from Theorem 1.1. Suppose that the conclusion holds for all integers $1 \leq m < k$. Now we prove that an arbitrary matching M_0 of size k in G is contained in a perfect matching of G.

Firstly we assume that $|M_0 \cap M| \geq 1$. Let $e \in M_0 \cap M$ and the vertex in D corresponding to e be v_e. Let $G' = G - V(e)$, $D' = D - v_e$, and $M' = M \backslash e$. Then D' is $(k-1)$-strong and $D' = D(G', M')$. By the induction hypothesis, G' is $(k-1)$-extendable. Hence $M_0 \backslash \{e\}$, which is a matching of size $k-1$ in G', is contained in a perfect matching M' of G'. Then $M' \cup \{e\}$ is a perfect matching of G containing M_0.

Now we handle the case that $M_0 \cap M = \emptyset$. In this case, M_0 corresponds to an arc set A_0 of order k of D. The arcs in A_0 form some independent cycles and paths in D. Let the set of cycles formed be $\mathrm{C}_0 = \{C_0, C_1, \ldots, C_{s-1}\}$ and the set of paths formed be $\mathrm{P}_0 = \{P_0, P_1, \ldots, P_{t-1}\}$. Let the starting and ending vertices of P_i be u_i and v_i, $0 \leq i \leq t-1$. Let V_0 be the union of the set of vertices of cycles in C_0 and the set of internal vertices of paths in P_0. Then $|V_0| = k - t$. By definition, $D - V_0$ is t-strong. By Lemma 2.3, there are t independent paths in D starting at v_i, $0 \leq i \leq t-1$, and ending at u_j, $0 \leq j \leq t-1$. Such paths, together with the paths in P_0, form some independent cycles in D. Denote the set of such cycles by C_1. Then $\mathrm{C}_0 \cup \mathrm{C}_1$ is a set of independent cycles in D which covers all arcs in A_0. $\mathrm{C}_0 \cup \mathrm{C}_1$ corresponds to a set C of independent M-alternating cycles in G. Let the set of edges of cycles in C be $E(\mathrm{C})$, then $E(\mathrm{C}) \triangle M$ is a perfect matching of G containing M_0. Hence G is k-extendable.

Conversely, suppose that G is k-extendable. To see that D is k-strong, let $\{v_1, v_2, \ldots, v_{k-1}\}$ be a set of $k-1$ vertices in D. Denote by e_i the edge in G corresponds to v_i, $1 \leq i \leq k-1$. Let $G' = G - \cup_{i=1}^{k-1} V(e_i)$, $D' = D - \{v_i : 1 \leq i \leq k-1\}$ and $M' = M \backslash \{e_i : 1 \leq i \leq k-1\}$. Then $D' = G(G', M')$. Since G is k-extendable, G' is 1-extendable. Hence D' is strong by Theorem 1.1 and D is k-strong.

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Theorem 2.4. *Let G be a bipartite graph and M a perfect matching of G. If G is minimal k-extendable then D = D(G,M) is minimal k-strong.*

Proof. Suppose that G is minimal k-extendable. By Theorem 2.1, D is k-strong. Let a be an arc of D and e be the edge corresponding to a in G. Then $D - a = D(G - e,M)$. By the minimality of G, $G – e$ is not k-extendable, hence $D - a$ is not k-strong by Theorem 2.1. By the arbitrary of a, D is minimal k-strong.

The converse of Theorem 2.4 does not generally hold, that is, G does not need to be minimal k extendable if $D = D(G,M)$ is minimal k-strong. For example, we show a minimal strong digraph D_0 in Figure 1 and $G_0 = B(D_0)$, which is not minimal 1-extendable, in Figure 2.

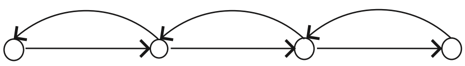

Figure 1: A minimal strong digraph D_0

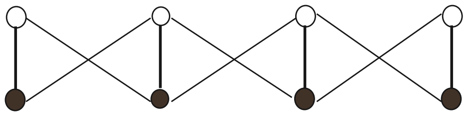

Figure 2: $G_0 = B(D_0)$

There are many parallel results on k-extendable bipartite graphs and k-strong digraphs. Theorem 2.1 and Theorem 2.4 help to explain such a similarity between these two classes of graphs. In the rest of this section, we will illustrate some such results.

Our first demonstrations are the well-known ear decompositions of strong digraphs and 1-extendable bipartite graphs.

An *ear decomposition* of a digraph D is a sequence E = $\{P_0,P_1,\ldots,P_t\}$, where P_0 is a cycle and each P_i is a path, or a cycle with the following properties:

(a) P_i and P_j are arc disjoint when i 6= j.

(b) For each $i = 1,\ldots,t$, if P_i is a cycle, then it has precisely one vertex in common with $V(D_{i-1})$. Otherwise the end-vertices of P_i are distinct vertices of $V(D_{i-1})$ and no other vertex of P_i belongs to $V(D_{i-1})$. Here D_i denotes the digraph with vertices $\bigcup_{j=0}^{i} V(P_j)$ and arcs $\bigcup_{j=0}^{i} A(P_j)$.

(c) $\bigcup_{j=0}^{t} V(P_j) = V(D)$ and $\bigcup_{j=0}^{t} A(P_j) = A(D)$.

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Conclusion:

Conversely, let D_1 be a strong component of D. Then $G_1 = B(D_1)$ is 1-extendable. To prove that G_1 is an elementary component or consist of a fixed double edge, we need only to prove that an edge $e = u_1u_2 \in E(G)\backslash E(G_1)$ associated with a vertex $u_1 \in V(G_1)$ is a fixed single edge. Suppose that e is not a fixed single edge and contained in a perfect matching M' of G. Let $u_1w_1, u_2w_2 \in M$, which correspond to vertices v_1 and v_2 in D respectively, then $v_1 \in V(D_1)$ and $v_2 \notin V(D_1)$. $M \triangle M'$ consists of nonadjacent edges and alternating cycles. The edges e, u_1w_1 and u_2w_2 must be contained in an alternating cycle C. However C corresponds to a directed cycle in D, which contains v_1 and v_2. This contradicts the fact that D_1 is a strong component of D.

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