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GRACEFUL LABELLING FOR CERTAIN GRAPHS

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Abstract

The study of famous K¨onigsberg bridge problem have been enunciated by L. Euler in 1736 and he has invented the technique to solve complicated problems. It directly pose a concern to such problem. After it no more efforts were made in the field of graph theory for about hundred years. In 1847 G. R. Kirchhoff enhanced the theory of trees for the applications in electrical circuit and networks. Ten years later, A. Cayley have found trees while he was trying to set out the isomers of hydrocarbons. The explosion of the computer age accelerated the growth of graph theory. Graph theory has a wide range of applications in computer science, electrical networks, and social sciences. Graphs have been shown to be an useful mathematical tool for explaining the structure of molecules in biology and chemistry. In one of his lectures in 1840, A. F. Mobius introduced the famous four-color dilemma. A. De Morgan addressed the issue with his colleagues mathematicians in London a decade later. De Morgan's explanation is credited as being the first systematic representation of the four-color issue. This issue has spurred graph theory research. The well-known four-color issue took a hundred years to solve. This issue was

addressed in 1976 by W. Haken and K. Appel. They've created an artificial map in which four colours aren't enough to depict separate regions with different hues.

Fundamental Concepts and Terminology

This chapter introduces the subject matter which are useful for the research work. Basic definitions are explained in this chapter with sufficient illustrations. Figures are also shown in this chapter.

2.2 Fundamental Concepts

Definition 2.2.1: Graph

A graph G = (V, E) consists of two finite sets: V , the vertex set of the graph which is a nonempty set of element called vertices, and E, the edge set of the graph which is a possibly empty set of element called edges, such that each edge e in E is assigned an unordered pair of vertices (u, v), called the end vertices of e.

Definition 2.2.2: Order of a Graph

The number of vertices in G is called the order of a graph G. It is denoted by |V (G)|.

Definition 2.2.3: Size of a Graph

The number of edges in G is called the size of a graph G. It is denoted by |E(G)|.

Definition 2.2.4: Loop

An edge of a graph that joins a vertex to itself is called a loop. A loop at the vertex v_i is an edge e = (v_i, v_i).

Definition 2.2.5: Multiple edges

If two vertices of a graph are joined by more than one edge then these edge are called multiple edges.

Definition 2.2.6: Simple Graph

A graph which has neither loops nor multiple edges is called a simple graph.

Definition 2.2.7: Adjacent Vertices

If two vertices of a graph are joined by an edge then these vertices are called *adjacent vertices*.

Definition 2.2.8: Degree of Vertices

The number of edges incident on vertex v of any graph G is called degree of v. It is denoted by $d_G(v)$ *or* $d(v)$.

Definition 2.2.9: Incident edges

Two edges that have an end vertex in common are called incident edges.

Definition 2.2.10: Endpoint/Pendent Vertex

A vertex of a graph of degree 1 is called endpoints or pendent vertex. An edge of the graph G which is incident with a pendent vertex is called a pendent edge

Definition 2.2.11: Connected and Disconnected Graph

A graph is said to be connected if their is a path between every pair of vertices of G. A graph which is not connected is called *a disconnected graph.*

Definition 2.2.12: Walk

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A walk is defined as a finite alternating sequence of vertices and edges of the form $v_0e_1v_1e_2 \ldots e_nv_n$ which start and end a vertex and each edge in the sequence is incident on the vertex immediately preceding and succeeding it in the sequence.

Definition 2.2.13: Path

A walk in which no vertex is repeated is called a path. A path with n vertices is denoted by P_n.

Definition 2.2.14: Trail

A walk in which no edge is repeated is called a trail.

Definition 2.2.15: Cycle

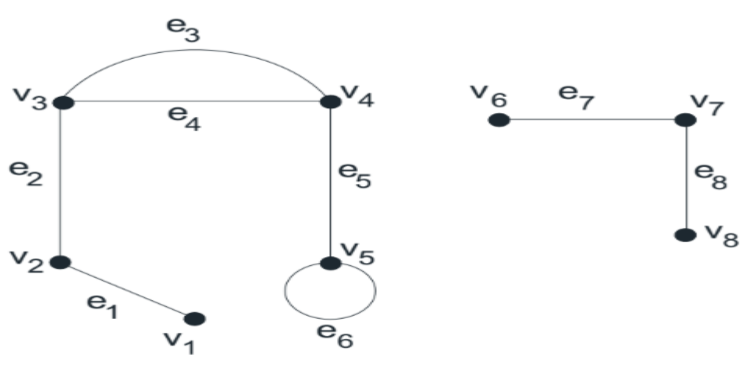
A closed path in which no vertex is repeated except the terminal vertex is called a cycle. A cycle with n vertices is denoted by C_n. The number of edges in a walk is called the length of the walk.

Definition 2.2.16: Unicyclic Graph

A graph G with exactly one cycle is called a unicyclic graph.

Beginning and ending vertices are equal then it is called a *closed walk*.

Illustration 2.2.17 : Let us consider the following graph G.



figure−2.1

In graph G shown in figure−2.1

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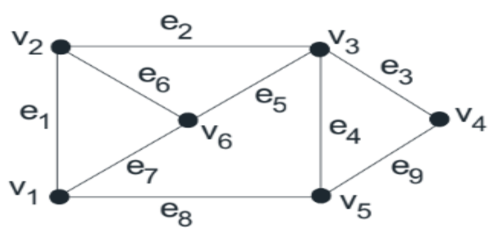
- Order of graph G is 8 and size of graph G is 8.
- e_6 forms a loop at v_5, e_3 and e_4 are multiple edges.
- v_1 and v_2 are adjacent vertices by e_1 edge.
- $d(v_5) = 3$, $d(v_3) = 2$, $d(v_6) = 1$.
- e_1 and e_2 are incident edges at v_2 vertex.
- v_1, v_6 and v_8 are endpoints.
- G is disconnected graph.

Illustration 2.2.18 : Consider the following graph G.



figure−2.2

For the figure−2.2 we note the following:

- $W = v_2e_2v_3e_3v_4e_9v_5e_4v_3e_5v_6e_6v_2$ is a closed walk.
- $P_4 = v_2e_2v_3e_3v_4e_9v_5$ is a path of length 3.
- $C_5 = v_2e_2v_3e_3v_4e_9v_5e_8v_1e_1v_2$ is a cycle of length 5.

Definition 2.2.19: Euler graph

Let G = (V, E) be a graph. A closed trail in G is called an Euler line if it contains all the edges of the graph G. A graph G is called an Euler graph if it admits an Euler line.

Various Techniques of Graph Labeling

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In the previous chapter, we have mentioned the fundamental concepts and terminology of graphs. This chapter is aimed to discuss cordial labeling of graphs in detail. For occurrence the problems arise from coloring of the vertices of a graph remained unsolved for more than ten decades for its solution in 1976. The problem of catalogue of isomers in the hydrocarbon series C_nH_{2n+2} initiated by the first work of Cayley is as old as the map coloring problem. In recent times, new contexts have emerged wherein the labeling of the vertices or edges of a given graph by elements of certain subsets S of the set of real numbers R. This problem provided enough motivation to formulate more terse mathematical problems on graph labelings. The concept of β-valuation was introduced by Alexander Rosa [6] in 1967. Independent discovery of β-valuation termed as graceful labeling by S.W.Golomb [7] in 1972, who also pointed out the importance of studying graceful graphs in trying to settle another complex problem of decomposing the complete graph by isomorphic copies of a given tree of the same order.

The most popular Ringel-Kotzig-Rosa [6] conjecture and various attempts to settle it provided the reason for different ways for labeling of graph structures. This chapter is focused on graceful and graceful like labelings of graphs.

3.2 Labeling of Graph

If the vertices are assigned values subject to specified condition in a graph, then it is known as graph labeling. Most interesting graph labeling problems have following three important rules. A set of numbers from which vertex labels are chosen. A rule that assigns a value to each edge. A condition that these values must satisfy. Now discussion about various graph labeling techniques will be carried out in sequential order as they were introduced.

3.3 Various Techniques of Graph Labeling

3.3.1 : Magic Labeling

Magic labeling was introduced by Sedl`acˇek [8] in 1963 motivated through the notion of magic squares in number theory. A function f is called magic labeling of a graph G if $f : V \cup E \longrightarrow \{1,$

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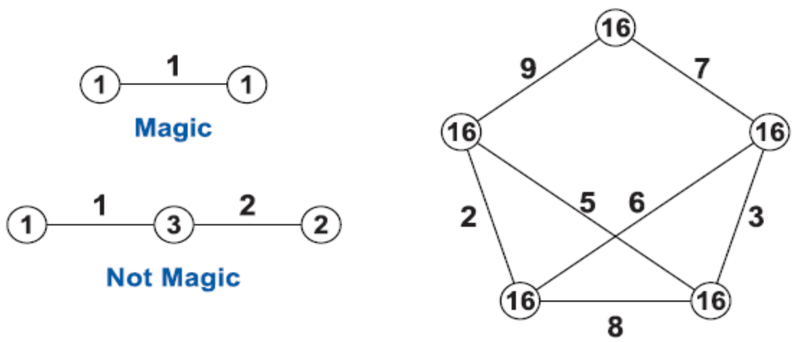
2, . . . , p + q} is bijective and for any edge *e = (u, v)*, the value of *f(u) + f(v) + f(e)* is constant. A graph which admits magic labeling is called magic graph.



figure–3.1

Some known results about magic labeling are listed.

Stewart [9] proved that

• K_n is magic for n = 2 and all n ≥ 5.

• $K_{n,n}$ is magic for all n ≥ 3.

• Fans F_n are magic if and only if n ≥ 3 and n is odd.

• Wheels W_n are magic for all n ≥ 4.

For any magic labeling *f* of graph *G*, there is a constant such that for all edges $e = (u, v) \in G$, $f(u) + f(v) + f(e) = c(f)$. The magic strength *m(G)* is defined as the minimum of *c(f)*, where the minimum is taken over all magic labeling of *G*.

The above definition and some facts listed below. They were obtained given by S. Avadyappan et. al. [10].

• $m(P_{2n}) = 5n + 1$, $m(P_{2n+1}) = 5n + 3$,

• $m(C_{2n}) = 5n + 4$, $m(C_{2n+1}) = 5n + 2$,

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• $m(K_{1,n}) = 2n + 4$.

Hegde and Shetty [11] defined $M(G)$ analogous to $m(G)$ as follows:

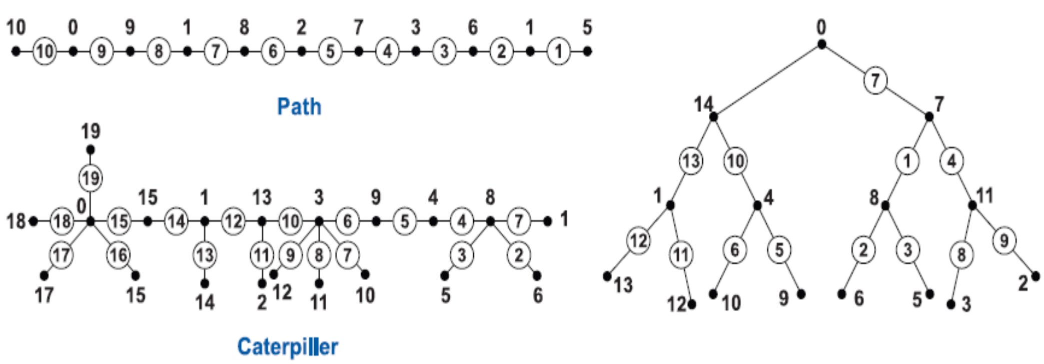
$M(G) = max\{c(f)\}$, where maximum is taken over all magic labeling f of G.

For any graph G with p vertices and q edges following inequality holds:

$p + q + 3 \leq m(G) \leq c(f) \leq M(G) \leq 2(p + q)$.

3.3.2 Graceful Labeling

A function f is called graceful labeling of a graph $G = (V, E)$ if $f : V \longrightarrow \{0, 1, . . . , q\}$ is injective and the induce function $f^{*} : E \longrightarrow \{1, 2, . . . , q\}$ defined as $f^{*}(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$. A graph which admits graceful labeling is called graceful graph. Initially Rosa named above defined labeling as β−valuation. Golomb [7] renamed β−valuation as graceful labeling. We will discuss graceful labeling in detail in Chapter 4.



figure−3.2

3.3.3 Graceful Like Labeling

In 1967 Rosa [6] gave another result of graceful labeling

A function f is called graceful like labeling of a graph $G = (V, E)$ if f : V → {0, 1, . . . , q + 1} is injective and the induce function $f^{*} : E \rightarrow \{1, 2, . . . , q - 1, q + 1\}$ defined as

$f^*(e) = |f(u) - f(v)|$ is bijective for every edge $e = (u, v) \in E$.

Some known results of Graceful-like labeling are mentioned below.

• Frucht [12] has shown that $P_m \cup P_n$ admits graceful like labeling with edge labels $\{1, 2, . . . , q - 1, q + 1\}$. $G \cup K_2$ (where G is graceful graph) admits graceful like labeling.

• Seoud and Elsahawi [13] have proved that all cycles C_n admit graceful like labeling.

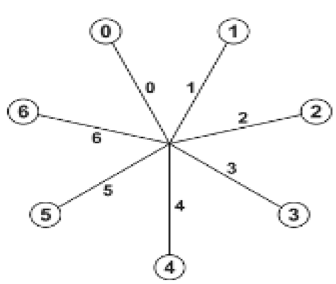
• Barrientos [14] proved that cycle C_n is having graceful like labeling with edge labels $\{1, 2, . . . , q - 1, q + 1\}$ if and only if $n \equiv 1$ or $2 \pmod 4$

3.3.4 Harmonious Labeling

Graham and Sloane [15] introduced harmonious labeling in 1980. They have introduced this during their study of modular versions of additive bases problems stemming from error correcting codes.

A function f is called Harmonious labeling of a graph $G = (V, E)$ if $f : V \to \{0, 1, . . . , q - 1\}$ is injective and the induce function $f^* : E \to \{1, 2, . . . , q - 1\}$ defined as $f^*(e) = (u, v) = (f(u) + f(v)) \bmod q$ is bijective.

A graph which admits harmonious labeling is called harmonious graph. We will demonstrate harmonious labeling by means of following examples in figure 3.3.



figure−3.3

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Graham and Sloane observed that if graph G is a tree then exactly two vertices are assigned same vertex label in harmonious labeling. Some known results about harmonious graph are listed below.

• Liu and Zhang [16] proved that every graph is a subgraph of a harmonious graph.

• Graham and Sloane [15] posed a conjecture Every tree is harmonious. In connection of above conjecture, Alderd and Mckay [17] proved that trees with 26 or less vertices are harmonious. They also proved that

– Caterpillars are harmonious.

– Cycles C_n are harmonious if and only if $n \equiv 1,3 \pmod 4$.

– Wheels W_n are harmonious for all n.

– $C_m \times P_n$ is harmonious if n is odd.

– K_n is harmonious if and only if $n \leq 4$.

– $K_{m,n}$ is harmonious if and only if m or n = 1.

– Fans F_n are harmonious for all n.

• Liu [16] proved that all helms are harmonious.

• Jungreis and Reid [18] proved that grids $P_m \times P_n$ are harmonious if and only if (m, n) 6= (2, 2). In the same paper they proved that $C_m \times P_n$ is harmonious if m = 4 and $n \geq 3$.

• Gallian et al. [19] proved that $C_m \times P_n$ is harmonious if n = 2 and m 6= 4.

DOUBLE PATH UNION AND ITS α-LABELING

In 1966 Rosa [6] defined α-labeling as a graceful labeling with an additional property that there is an integer k *(0 ≤ k < |E(G)|)* such that for every e = (x, y) ∈ E(G), either *f(x) ≤ k < f(y) or f(y) ≤ k < f(x).* A graph which admits α-labeling is necessarily bipartite graph with partition of V (G) = $V_1{}^S V_2$, where $V_1 = \{v \in V(G)/f(x) \leq k\}$ and $V_2 = \{v \in V(G)/f(x) > k\}$. We call such graph *G* with α-labeling *f* as an α-graceful graph.

In [35] Kaneria, Viradia and Makadia proved that the path union of a semi smooth graceful graph, star of a semi smooth graceful graph and cycle graph of a semi smooth graceful graph are graceful. They also proved step grid graphs St_n, Cycle of graphs $C(t \cdot H)$ *and* $C^m(t \cdot C_n)$ are smooth graceful graphs, where t ≡ 0 (mod 4), n ≡ 0 (mod 4), m ∈ N and H is a semi smooth graceful graph. Some of these results we discuss here and we also discuss equivalentness of α-labeling and semi smooth graceful labeling. We also derived α-labeling for double path union of some graph.

CONCLUSION

Graph labeling is becoming more interesting filed due to its broad range of applications. A vital role have played by labeled graphs in various fields of graph theory. Coding theory, missile guidance codes, design of good Radar type codes, astronomy, circuit design, X-ray crystallography, data base management are few names of such important fields. This chapter gives an overview of graph labeling as well as some information of important applications.

Here we would like to enhance the graph labeling applications in the filed of computer science. Graph labeling applications have been studied and here we explore the usage of this field in several area like communication networks, image processing, data mining, crypto systems and bird view has been proposed. Graph theory has been applied in investigation of electrical network is a collection of components and device interconnected electrical gazettes.

The network components are idealized physical devices and system, in order to represent several properties. Also they must obey the Kirchhoff's law of currents and voltage.

5.2 Bipartite Graph and Time Table Scheduling

Allocation of classes and subject to all the teachers in an institute is one of major issues, whenever constrain and complexity occur. A bipartite graph helps to solve such problem. Also it play an important role in this kind of problems. For m teachers and n subjects available periods p, the time table has to be prepared as follow. A bipartite graph G, we mean a set of teachers v_1, v_2, . . . , v_m and another set of subjects u_1, u_2, . . . , u_n. These vertices have p_i periods. It is presumed that any one period, each teacher may engage almost one subject. Also each subject can be taught by maximum one teacher. For the first period, the time table for this single period correspond to a matching in the bipartite graph G and conversely, each matching correspond to a possible assignment of some teachers to subject taught during that period.

Hence, the solution for this will be obtained by partitioning the edges of the given graph into minimum number of matching. Also the edge have to be colored with minimum number of colors and this problem can be solved by the vertex coloring algorithm. The line graph of given graph has equal number of vertices and edges of the given graph. Also the vertices in the line graph are adjacent iff they are incident in the given graph. The line graph is a simple graph and its proper coloring gives a proper edges coloring of the given graph.

REFERENCES

[1] M.E. Abdel-Aal, New families of odd harmonious graphs, International journal of soft computing, mathematics and control (IJSCMC), Vol.3, No.1.(2014).

[2] B.D. Acharya and S.M.Hedge, Arithemetic graphs, J.Graph Theory, 14(3),(1990) , 275-299.

[3] V. Ajitha, K.L. Princy, V. Lokesha, P.S. Ranjini, On square difference graphs, International Journal of Math. Combin., 1, (2012), 31-40.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

[4] Binthiya R, Sarasija P.B., Some new Even Harmonious Graphs, International Journal of Mathematics and Soft Computing, Vol.4, No.2, (2014), 105-111.

[5] J.A. Bondy and U.S.R.Murthy, Graph Theory with Applications, Macmillan, London, (1976).

[6] Dushyant Tanna (2013), 'Harmonious Labeling on certain Graphs', International Journal of Advance Engneering Research and Studies.46-48.

[7] B. Gayathri and M. Subbiah, Edge graceful labeling of some trees, Global Journal of Mathematical Sciences : Theory and Practical, Vol .3, No.1, (2011), 27 – 33.

[8] B. Gayathri and M. Subbiah ,Strong edge graceful labelings of some graphs, Bull . Pure Appl .Sci., 27E (2008), 89-98.

[9] B.D. Gemina and T.M.K. Anandavally, Some new perspectives on arithmetic Graphs, Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi (2008): 41- 46.

[10] R.B. Gnanajothi, Topics in Graph Theory, Ph.D Thesis, Madurai Kamaraj University, (1991).

[11] R.L. Graham and N.J.A. Sloane, On additive bases and harmonious graphs, SIAM J. Alg. Discrete Meth.,1(1980) 382 - 404.

[12] Gusti A. Saputri, Kiki A. Sugeng, Dalibor Froncek, The Odd Harmonious Labeling of Dumbbell and Generalized Prism Graphs, AKCE Int. J. Graphs Comb., 10, No.2, (2013), 221-228.

[13] F. Harary, Graph Theory, Narosa Publishing House Reading, New Delhi, (2001).

[14] S. M. Hedge, and Sudhakar Shetty, On Arithemetic graphs, J. Pure appl Math., 33(8), (2002), 1275-1283.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

[15] S.M. Hegde and Sudhakar Shetty, Combinatorial Labelings of Graphs, Applied Mathematics E-Notes, 6 (2006), 251-258.

[16] P. Jeyanthi , S. Philo, Odd Harmonious Labeling of Some New families of Graphs. Electronic Notes in Discrete Mathematics 48, (2015), 165-168.

[17] M.I. Jinnah and G.S. Singh, A note on arithmetic numberings of graphs, Proc. Symposium on Graphs and combinatorics, Kerala, india (1991). 83 - 87.

[18] Joseph A. Gallian, A dynamic survey of graph labeling, The Electronic Journal of Combinatorics (2014).

[19] Joseph A. Gallian and Lori Ann Schoenhard, Even harmonious graphs, AKCE International Journal of graphs comb., Vol 11, No.1 (2014), 27-49.

[20] Z - H Liang and Z-L Bai , On the odd harmonious graphs with applications, J.Appl. Math. Comput., 29(2009) ,105-116.

[21] B.Liu and X.Zhang, On harmonious labeling of graphs, Ars. Combin., 36(1993) ,315 – 326.

[22] S.P. Lo, On edge graceful labeling of graphs, Congress Numerantium, 50 (1985) 231-241.

[23] Martin Baca and Maged Z. Youssef , On Harmonious Labeling of Corona Graphs , Journal of Applied Mathematics , Volume 2014 , 2014 , Article ID 627248, 4 pages.

[24] M.A. Perumal, S. Navaneethakrishnan, A. Nagarajan, S. Arockiaraj, Super graceful labeling for some simple graphs, International Journal of Mathematics and Soft Computing , Vol 2, No.1, (2012) , 35 – 49.

[25] Rosa A, 'On certain valuations of the vertices of a graph,' Theory of Graphs (Internat. symposium, Rome, July 1966), Gordon and Breach, N. Y. and Dunod Paris, (1967), 349- 355.