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Research Paper

Equitable Edge Coloring of a Prism and Generalized Petersen Graphs $\boldsymbol{P(n,m)}$ for any $\boldsymbol{n}$ and $\boldsymbol{m = 2, 3}$

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Abstract

An Equitable coloring of graph G is a proper k -coloring C that verifies the following property: for every color class $c_i, 1 \leq i \leq k$, there exists a vertex x_i, with color c_i, such that all the other colors in C are utilized in x_i neighbors.

In this paper, we discussed the equitable coloring of the prisms and generalized Petersen graphs $P(m, n)$,

Keywords: Prism graph; Generalized Petersen graph.

Introduction

A prism Y_m^n is a simple graph and is obtained as the cartisian product of the cycle C_m and the path P_{n-} The prism Y_m^n has mn vertices and $m(2n-1)$ edges. Coxeter [5] introduced the generalization of Petersen graphs.

The construction of the prism with n -layers Y_n^m by considering the cartesian product of a cycle C_m and a path P_n is given in Gallian.

We have considered the prism for Y_n^2 as Y_n leaving 2. The generalized Petersen graph family was introduced by Coxeter [12] and these graphs were given their name by Mark Watkins in 1969. Coxeter defined the generalized Petersen graph $P(n, m)$ for the natural numbers n and $m, n > 2m$. Naserasr et al. proved the Petersen graph is not 3 edge colorable in 2003 , Babak et al. gave the minimum vertex cover of generalized Petersen graphs in 2010 , Davtyan gave the parameters of the Petersen graphs in 2013 and Sudha et al. gave the equitable colorings of prisms and generalized

The following upper bound for an equitable coloring of a graph, presented in Gallian, has been proved to be very useful. If G admits an equitable coloring of graph with m colors, then G must have at least m vertices with degree at least $m-1$. The m -degree of a graph G, denoted by $m(G)$, is the largest integer m such that G has m vertices of degree at least $m-1$ For a given araph G. it mav be easilv remarked that $\gamma(G) < \varphi(G) < m(G)$

Preliminaries

we have found the equitable edge coloring of a prism Y_n and its equitable edge chromatic number to be 3 for all n.

Moreover, we have also discussed the equitable edge coloring of the generalized Petersen graphs for the following cases:

(i) the generalized Petersen graph $P(n, 2), n > 5$ and

(ii) the generalized Petersen graph $P(n, 3), n > 7$.

Theorem 1.1. The prism Y_n for $n > 2$ admits equitable edge coloring and its chromatic number is 3

Proof. Let the cycle C_n has the vertex set $\{v_i/1 \leq i \leq n\}$ and the edge set $\{v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{v_n v_1\}$.

The cartesian product of the cycle C_n with the path P_2 is a prism Y_n. The vertex set of Y_n is

$$V(Y_n) = \{v_i/1 \leq i \leq n\} \cup \{u_i/1 \leq i \leq n\}$$

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where,

(i) $\{v_i/1 \leq i \leq n\}$ represent the inner vertices and

(ii) $\{u_i/1 \leq i \leq n\}$ represent the outer vertices.

and the edge set of Y_n is

$$E(Y_n) = \{v_i v_{i+1}/1 \leq i \leq n-1\} \cup \{v_n v_1\} \\ \cup \{u_i u_{i+1}/1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{v_\mathrm{i} u_\mathrm{i}/1 \leq i \leq n\}.$$

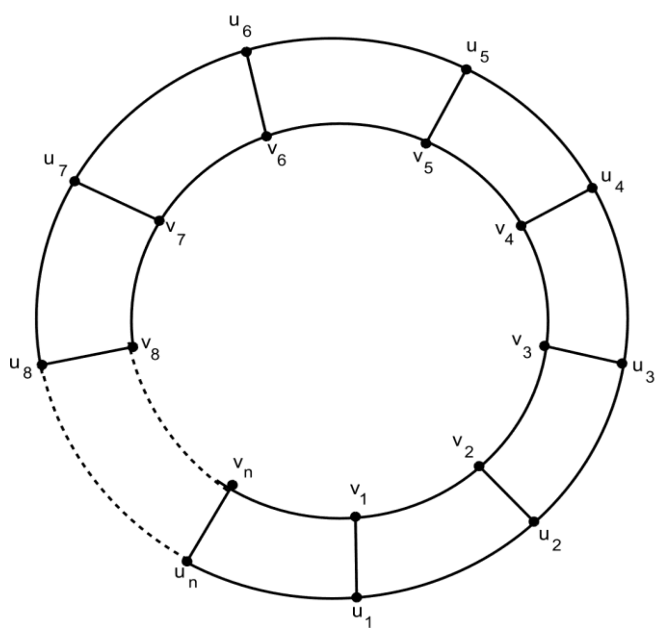

Figure 1; 1: Prism Y_m

There are two cases :

(i) n is even

(ii) (ii) n is odd

Define the function f from the edges of Y_n to the color set $\{1,2,3\}$ as follows :

Case (i): Let n be even.

For $1 \leq i \leq n-1$, $$f(u_\mathrm{i} u_{\mathrm{i}+1}) = \begin{cases} 2 & \text{if } i \equiv 1(\mathrm{mod}2) \\ 3 & \text{if } i \equiv 0(\mathrm{mod}2) \end{cases}$$
$$f(u_n u_1) = 3$$

for $1 \leq i \leq n$,

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With this type of coloring Y_n satisfies the definition of edge coloring and $|C|1]| = |C[2]| = |C[3]\ |= n$

Hence the prism Y_n admits equitable edge coloring and its chromatic number of equitable edge coloring of Y_n for even n is 3

Case (ii): Let n be odd.

For $1 \leq i \leq n-1$

$$\begin{aligned} f(u_i u_{\mathrm{i}+1}) &= \begin{cases} 2 & \text{if } i \equiv 1(\text{mod}2) \\ 3 & \text{if } i \equiv 0(\text{mod}2) \end{cases} \\ f(u_n u_1) &= 1 \end{aligned}$$

for $2 \leq i \leq n-1$

$$f(u_i v_i) = 1$$

$$f(u_1 v_1) = 3$$
$$f(u_n v_n) = 2,$$

and for $1 \leq i \leq n-1$

$$\begin{aligned} f(v_{\mathrm{i}} v_{i+1}) &= \begin{cases} 2 & \text{if } i \equiv 1(\text{mod}2) \\ 3 & \text{if } i \equiv 0(\text{mod}2) \end{cases} \\ f(v_n v_1) &= 1 \end{aligned}$$

With this type of coloring Y_n satisfies the definition of edge coloring and $|C[1]| = |C[2]| = |C[3]| = n$

Hence the prism Y_{n} admits exuitable edge coloring and its chromatic number of equitable edge coloring of Y_n for odd n is 3 .

Illustration 1.1. Consider the prism $\boldsymbol{Y_s}$.

Using theorem 2. 1, case (i) for even n, assign the color 1 to the edges u_1v_1, u_2v_2 $u_3v_3, u_4v_4, u_6v_6, u_6v_6, u_7v_7, u_8v_8$, the color 2 to the edges $u_1u_2, u_3u_4, u_5u_6, u_7u_{\mathrm{g}}, v_1v_2$ v_3v_4, v_6v_6, v_7v_8 and the color 3 to the edges $u_3, u_4u_6, u_6u_7, u_8u_1, v_2v_3, v_4v_6, v_6v_7, v_8v_1$ as shown in figure 2.2.

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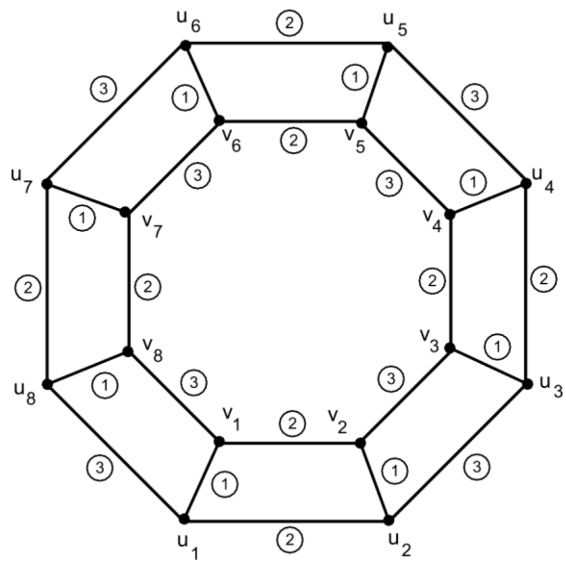

Figure 1.2: the prism Y_s

Here $|C[1]| = |C[2]| = |C[3]| = 8$ and satisfy the condition $|(|C[i]| - |C[j]|)| < 1$ for $1 \leq i \leq 3;\ 1 \leq j \leq 3$. Hence $X_{e\,e}(Y_8) = 3$.

Illustration 1.2. Consider the prism Y_5.

Using theorem 1.1, case (ii) for odd n, assign the color 1 to the edges $u_5u_1;\ v_5v_1;\ u_2v_2;\ u_3v_3;\ u_4v_4$, the color 2 to the edges $u_1u_2;\ u_3u_4;\ v_1v_2;\ v_3v_4;\ u_5v_5$ and the color 3 to the edges $u_2u_3;\ u_4u_5;\ v_2v_3;\ v_4v_5;\ u_1v_1$ as shown in figure 1.3.

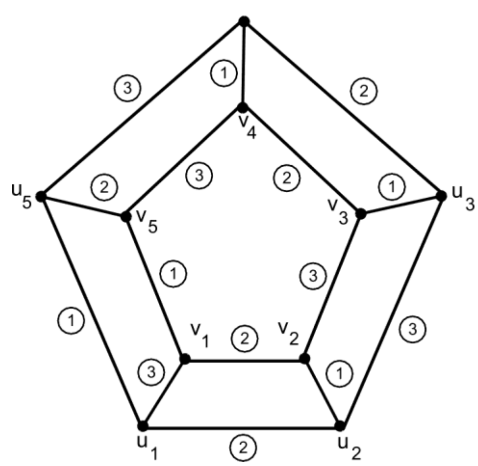

Figure 1.3: Prism Y_5

Here $|C[1]| = |C[2]| = |C[3]| = 5$ and satisfy the condition

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$$|(|C[i]| - |C|j||)| \leq 1 \text{ for } 1 \leq i \leq 3, 1 \leq j \leq 3$$

Hence $\chi_{e\,e}(Y_5) = 3$.

Theorem 1.2. The generalized Petersen graph $P(n,2), n > 5$ admits equitable edge coloring and $\chi_{e\,e}(P(n,2)) = 3$

Proof. Let $P(n,2), n > 5$ be the generalized Petersen graph with the vertex set

$$V(P(n,2)) = \{u_i / 1 \leq i \leq n\} \cup \{v_i / 1 \leq i \leq n\}$$

and the edge set.

$$E(P(n,2)) = \{u_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{v_i v_{i+2} / 1 \leq i \leq n-2\} \cup \{v_{n-1} v_1, v_n v_2\} \cup \{u_i v_i / 1 \leq i \leq n\}$$

Theorem 1.3. The generalized Petersen graph $P(n,2), n > 5$ admits interval edge coloring and $\chi_{ie}(P(n,2)) = \delta(P(n,2))$.

Proof. Let $P(n,2), n > 5$ be the generalized Petersen graph with the vertex set

$$V(P(n,2)) = \{u_i / 1 \leq i \leq n\} \cup \{v_i / 1 \leq i \leq n\}$$

and the edge set

$$E(P(n,2)) = \{u_i u_{i+1} / 1 \leq i \leq n-1\} \cup \{u_n u_1\} \cup \{v_i v_{i+2} / 1 \leq i \leq n-2\} \cup \{v_{n-1} v_1, v_n v_2\} \cup \{u_i v_i / 1 \leq i \leq n\}$$

There are three cases:

(i) $n \equiv 0 (\text{mod} 4)$

(ii) $n \equiv 2 (\text{mod } 4)$

(iii) $n \equiv 1 (\text{mod} 4)$ or $n \equiv 3 (\text{mod} 4)$

Case (i) : Let $n \equiv 0(\text{ mod } 4)$.

The outer edges are colored is s

$$f(u_i u_{i+1}) = \begin{cases} 2 & \text{if } \varepsilon \equiv 1 (\text{mod} 2) \\ 3 & \text{if } \varepsilon \equiv 0 (\text{mod} 2) \end{cases}$$
$$\text{for } 1 \leq 1 \leq n-1$$
$$\text{and } f(u_n u_1) = 3.$$

The inner edges are colored as

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$$f(v_i v_{1+2}) = \begin{cases} 2 & \text{if } \varepsilon \equiv 1(\text{mod}4) \text{ or } t \equiv 2(\text{mod}4) \\ 3 & \text{if } \varepsilon \equiv 3(\text{mod}4) \text{ or } t \equiv 0(\text{mod}4) \end{cases}$$

$$\text{for } 1 \leq 1 \leq n-2,$$

$$f(v_{n-1}v_1) = 3$$

$$\text{and } f(v_n v_2) = 3$$

The edges $\{u_i v_t\}$ are colored ss

$$f(u_i v_t) = 1 \text{ for all } 1$$

The colors $\{1,2,3\}$ are used in ooloring $P(n,2), n \equiv 0(\text{mod}4)$ to satisfy the condition of interval edge coloring.

$$\text{Therefore } \chi_k(P(n,2)) = \delta(P(n,2)) \text{ for } n \equiv 0(\text{mod}4)$$

Case (ii) : Let $n \equiv 2(\text{mod}4)$.

The outer edges are colored is

$$f(u_1 u_{i+1}) = \begin{cases} 3 & \text{if } t \equiv 1(\text{mod}2) \\ 2 & \text{if } t \equiv 0(\text{mod}2) \\ & \text{for } 1 < 1 < \pi - 1; \\ f(u_1 u_2) & = 1 \\ f(u_{n-1} u_n) & = 1 \\ \text{and } f(u_n u_1) & = 3. \end{cases}$$

The inner edges are colored as

$$f(v_i v_{i+2}) = \begin{cases} & 3 \text{ if } t \equiv 1(\text{mod}4) \text{ or } 1 \equiv 0(\text{mod}4) \\ 2 & \text{if } t \equiv 2(\text{mod}4) \text{ or } 1 \equiv 3(\text{mod}4) \end{cases}$$

$$\text{for } 1 \leq 1 < n-1$$

$$f(v_{n-1}v_1) = 1$$
$$\text{and } f(v_n v_2) = 1$$

The edges $\{u_i v_4\}$ are colored as

$$f(u_4 v_i) = 1 \text{ for } 2 < 1 < n-1$$

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$$\begin{aligned} f(u_1v_1) &= 2, \\ f(u_2v_2) &= 3 \\ f(u_{n-1}v_{n-1}) &= 3 \\ \text{and } f(u_nv_n) &= 2. \end{aligned}$$

The colors $\{1,2,3\}$ are used in coloring. $P(n,2), n \equiv 2(\text{mod}4)$ to satisfy the condition of interval edge coloring.

Therefore $\chi_{iz}(P(n,2)) = \delta(P(n,2))$ for $n \equiv 2(\text{mod}4)$

Case (iii) : Let $n \equiv 1(\text{mod}4)$ or $n \equiv 3(\text{mod}4)$.

Type (a): Let $n = 7 + 12j, j = 0,1,2, \ldots$

The outer edges are colored as

$$f(u_iu_{i+1}) = \begin{cases} 2 & \text{if } i \equiv 1(\text{mod}2) \\ 3 & \text{if } i \equiv 0(\text{mod}2) \\ & \text{for } 1 \leq 1 < n-2, \end{cases}$$

$$f(u_{n-2}u_{n-1}) = 1$$

$$f(u_{n-1}v_n) = 2$$

$$\text{and } f(u_nu_1) = 1.$$

The inner edges are colored as

$$f(v_iv_{1+2}) = \begin{cases} 2 & \text{if } 1 \equiv 1(\text{mod}4) \text{ or } t \equiv 0(\text{ mod } 4) \\ 3 & \text{if } t \equiv 2(\text{mod}4) \text{ or } 1 \equiv 3(\text{mod}4) \end{cases}$$

$$\text{for } 1 \leq 1 < n-2,$$

$$f(v_{n-2}v_n) = 1$$

$$f(v_{n-1}v_1) = 1$$

$$\text{and } f(v_nv_2) = 2.$$

In case (iii) there are six subcases

(a) $n = 7 + 12j, j = 0,1,2, \ldots$

(b) $n = 9 + 12j, j = 0,1,2, \ldots$

(c) $n = 11 + 12j, j = 0,1,2, \ldots$

(d) $n = 13 + 12j, j = 0,1,2, \ldots$

(e) $n = 15 + 12j, j = 0,1,2, \ldots$

(f) $n = 17 + 12j, j = 0,1,2, \ldots$

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The function f is defined as before as in the proof of the theorem 1.3 from the edges of the generalized Petersen graph $P(n,2), n > 5$ to the set of colors $\{1,2,3\}$ for all the three cases.

The colors $\{1,2,3\}$ are used in coloring $P(n,2)$ to satisfy the condition of edge coloring. The color classes $C[1], C[2], C[3]$ satisfy the condition $|C[1]| = |C|||2 = |C|||3 = n$. Hence $P(n,2)$ satisfies the equitable edge coloring for all cases.

Therefore $\chi_{e\,e}\,(P(n,2)) = 3$ for $n > 5$.

Illustration 1.3. Consider the generalized Petersen graph $P(12,2)$.

Using theorem 1.2, case (i) for $n \equiv 0(\mathrm{mod}4)$, assign the color 1 to the edges $u_1v_1, u_2v_2, u_9v_3, u_4v_i, u_5v_5, u_6v_6, u_7v_7, u_8v_g, u_gv_9, u_{10}v_{10}, u_{11}v_{11}, u_{12}v_{12}$, the color 2 to the edges $u_1u_2, u_9u_4, u_6u_6, u_7u_s, u_9u_{10}, u_{11}u_{12}, v_1v_8, v_2v_4, v_5v_7, v_6v_8, v_yv_{11}, v_{10}v_{12}$ and the color 3 to the edges $u_2u_3, u_4u_5, u_6u_7, u_8u_9, u_{10}u_{11}, u_{12}u_1, v_3v_5, v_4v_6, v_7v_9, v_gv_{10}, v_1v_1.\ v_{12}v_2$ as shown in figure 1.4.

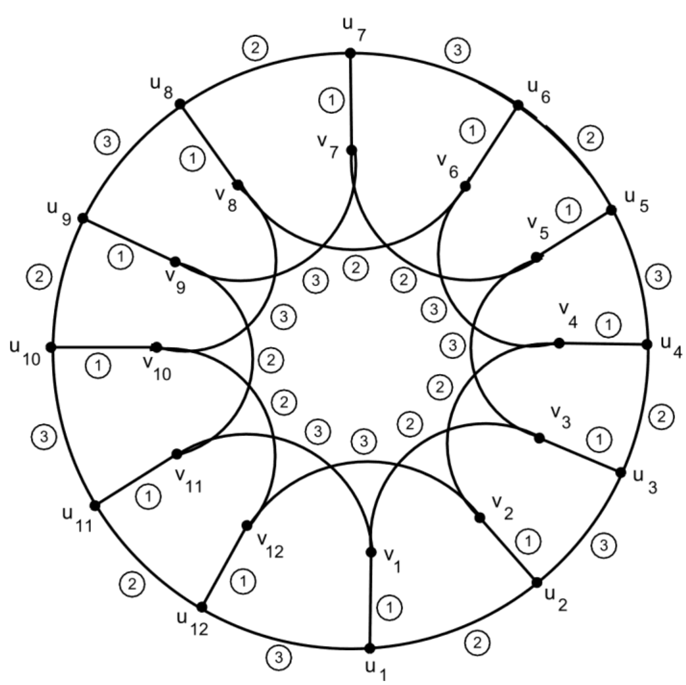

Figure 1.4: Generalized Petersen graph $P(12,2)$

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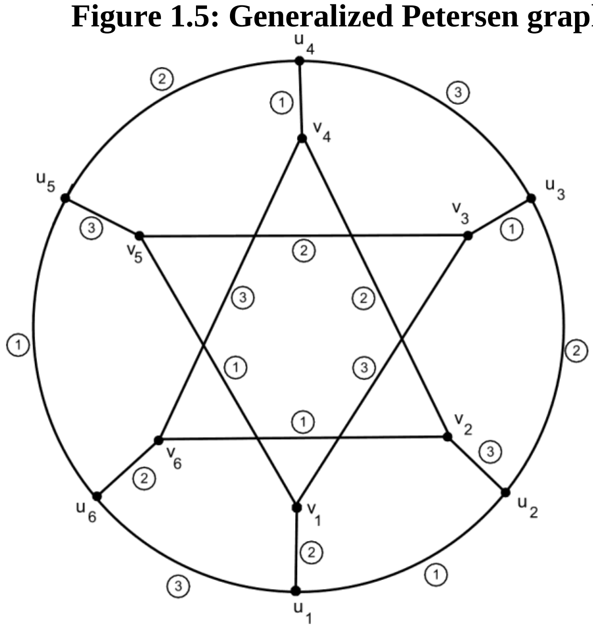
With this type of coloring, the generalized Petersen graph $P(12{,}2)$ satisfies the definition of the edge coloring.

Here $|C[1]| = |C|||2 = |C[3]| = 12$. Therefore $\chi_{e\,e}\,(P(12{,}2)) = 3$.

Illustration 1.4. Consider the generalized Petersen graph $P(6{,}2)$.

Using theorem 2.2, case (ii) for $n \equiv 2(\bmod 4)$, assign the color 1 to the edges $u_1u_2, u_6u_6, v_5v_1, v_5v_2, u_3v_3, u_4v_4$, the color 2 to the edges $u_2u_3, u_4u_5, v_2v_4, v_3v_5, u_1v_1$ u_6v_6 and the color 3 to the edges $u_3u_4, u_6u_1, v_1v_3, v_4v_6, u_2v_2, u_5v_5$ as shown in figure 1.5

Figure 1.5: Generalized Petersen graph $P(6,2)$



With this type of coloring, the generalized Petersen graph $P(6{,}2)$ satisfies the definition of the edge coloring.

Here $|C[1]| = |C|2]| = |C[3]\ |= 6$. Therefore $\chi_{e\ e}\,(P(6{,}2)) = 3$.

Illustration 1.5. Consider the generalized Petersen graph $P(19{,}2)$.

Using theorem 1.2, case (iii) (a) for $n = 7 + 12j, j = 0{,}1{,}2, \ldots$, assign the color 1 to the edges $u_{17}u_{18}, u_{19}u_1, v_{17}v_{19}, v_{18}v_1, u_2v_2, u_3v_{3,}u_4v_4, u_5v_5, u_6v_6, u_7v_7, u_8v_8, u_9v_9,$ $u_{10}v_{10}, u_{11}v_{11}, u_{12}v_{12}, u_{13}v_{13}, u_{14}v_{14}, u_{15}v_{15}, u_{16}v_{16}$, the color 2 to the edges u_1u_2, u_3u_4 $u_5u_6, u_7u_9, u_9u_{10}, u_{11}u_{12}, u_{13}u_{14}, u_{15}u_{16}, u_{18}u_{19}, v_1v_3, v_4v_6, v_5v_7, v_8v_{10}, v_gv_{11}, v_{12}v_{14}, v_{11}v_{15}$

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$v_{16}v_{18}, v_{19}v_2, u_{17}v_{17}$ and the color 3 to the edges $u_2u_3, u_4u_5, u_6u_7, u_8u_9, u_{10}u_{11}, u_{12}u_{13}$ $u_{14}u_{15}, u_{16}u_{17}, v_2v_4, v_3v_6, v_6v_8, v_7v_y, v_{10}v_{12}, v_{11}v_{13}, v_{14}v_{16}, v_{15}v_{17}, u_1v_1, u_{18}v_{18}, u_{19}v_{19}$ as shown in figure 1.6

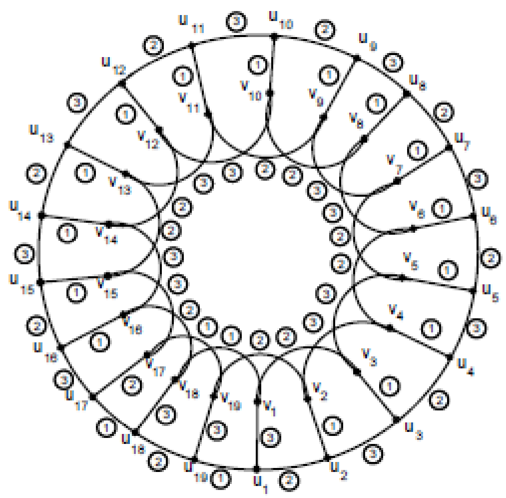

Figure 1.6: Generalized Petersen graph $P(19, 2)$

With this type of coloring, the generalized Petersen graph $P(19,2)$ satisfies the definition of the edge coloring.

Here $|C[1]| = |C|2]| = |C \mid 3] \mid = 19$.

Therefore $\chi_{e\,e}(P(19,2)) = 3$.

Illustration 1.6. Consider the generalized Petersen graph $P(33,2)$.

Using theorem 1.2, case (iii) (b) for $n = 9 + 12j, j = 0,1,2, \ldots$, assign the color 1 to the edges $u_3u_4, u_6u_7, u_9u_{10}, u_{12}u_{13}, u_{15}u_{16}, u_{18}u_{19}, u_{21}u_{22}, u_{24}u_{25}, u_{27}u_{28}, u_{30}u_{31}, u_{33}u_1$, $v_1v_3, v_4v_6, v_7v_9, v_{10}v_{12}, v_{13}v_{15}, v_{16}v_{18}, v_{19}v_{21}, v_{22}v_{24}, v_{2\pi}v_n, v_{2s}v_{30}, v_{31}v_{33}, u_2v_2, u_5v_5, u_8v_8$, $u_{11}v_{11}, u_{14}v_{14}, u_{17}v_{17}, u_{20}v_{20}, u_{23}v_{23}, u_{26}v_{25}, u_{29}v_{29}, u_{32}v_{32}$, the color 2 to the edges $u_2u_3u_5u_6, u_8u_9, u_{11}u_{12}, u_{14}u_{15}, u_{17}u_{18}, u_{20}u_{21}, u_{23}u_{24}, u_{26}u_{27}, u_{2j}u_{30}, u_{32}u_{33}, v_3v_5, v_6v_8, v_yv_{11}$ $v_{12}v_{14}, v_{15}v_{17}, v_{18}v_{20}, v_{21}v_{23}, v_{24}v_{25}, v_{27}v_{29}, v_{30}v_{32}, v_{33}v_2, u_1v_1, u_4v_4, u_7v_7, u_{10}v_{10}, u_{13}v_{13}$ $u_{16}v_{16}, u_{19}v_{19}, u_{22}v_{22}, u_{25}v_{25}, u_{25}v_{28}, u_{31}v_{31}$ and the color 3 to the edges u_1u_2, u_4u_5 $u_7u_g, u_{10}u_{11}, u_{13}u_{14}, u_{16}u_{17}, u_{19}u_{20}, u_{22}u_{23}, u_{25}u_{26}, u_{28}u_{2a}, u_{31}u_{32}, v_2v_4, v_5v_6, v_8v_{10}, v_{11}v_{13}$ $v_{14}v_{16}, v_{17}v_{19}, v_{20}v_{22}, v_{23}v_{25}, v_{26}v_{28}, v_{2y}v_{31}, v_{32}v_1, u_9v_3, u_6v_6, u_9v_3, u_{12}v_{12}, u_{15}v_{15}, u_{18}v_{18}$ $u_{21}v_{21}, u_{24}v_{2t}, u_{27}v_{27}, u_{30}v_{30}, u_{33}v_{33}$ as shown in figure 2.7.

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Here $|C[1]| = |C[2]| = |C[3]| = 11$.

Therefore $\chi_{ee}(P(11,3)) = 3$.

Conclusion:

The study of Equitable edge coloring of a prism and generalized Petersen graphs and other coloring diagrams are significant because of its applications in some genuine issues like bunching, programmed acknowledgment of reports, web administration and so forth In this paper, we researched a fair edge shading, chromatic number of crystal chart and Generalized Petersen diagram. The examination of comparable to results for various diagrams and diverse activity of above groups of chart are as yet open

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