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Hahn Banach Theorem and Its Applications

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Abstract

The Hahn-Banach theorem is one of the major theorems that we face in a first course on Functional Analysis. We have a very powerful collection of theorems with the Banach-Steinhaus theorem, the open mapping theorem, and the closed graph theorem when we combine them. The third, fourth, and fifth theorems all need the completeness of the spaces involved, whereas the Hahn-Banach theorem does not. It's available in two styles: analytic and geometric. The geometric version deals with the separation of disjoint convex sets using hyperplanes, whereas the analytic version deals with the extension of continuous linear functionals from a subspace to the full space with predefined properties. Both versions have applications outside of Functional Analysis, such as in optimization theory and partial differential equations theory, to mention a few.

The Hahn-Banach theorem and some of its applications will be discussed in this article. We will not provide extensive proofs of the main theorems because they are available in any functional analysis book

INTRODUCTION

One of the major theorems proven in the introductory course on Functional Analysis is the Hahn-Banach theorem. It has a wide range of applications, not just within the subject but also in other fields of mathematics such as optimization and partial differential equations. The Hahn-Banach theorem, its ramifications, and some applications will be discussed in this article.

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The Hahn-Banach theorems

The analytic and geometric versions of the Hahn-Banach theorem follow from a general theorem on the extension of linear functionals on a real vector space.

*(*Hahn-Banach Theorem*) Let V be a vector space over* R.

Theorem 1.1. Let p : *V* → R *be a mapping such that*

$$\left.\begin{array}{rcl} p(\alpha x) & = & \alpha p(x) \\ p(x+y) & \leq & p(x)+p(y) \end{array}\right\} \quad (1.1)$$

for all x and y ∈ *V and for all α* > 0 *in* R. *Let W be a subspace of V and let g* : *W* → R *be a linear map such that*

$$g(x) \leq p(x)$$

for all x ∈ *W. Then, there exists a linear extension f* : *V* → R *of g (i.e. f(x) = g(x) for all x* ∈ *W) which is such that*

$$f(x) \leq p(x)$$

for all x ∈ *V.*

Zorn's lemma is used to prove the aforementioned theorem. Let P denote the set of all pairs (Y, h), where Y is a subspace of V that includes W and h: Y R is a linear map that is an extension of g and also has the property of having the property of having the property of having the property of having the property of having the property of having the property of having the property of having the

$$h(x) \leq p(x)$$

for all $x \in Y$. Clearly P is non-empty, since $(W, g) \in P$. Consider the partial order defined on P by

$$(Y, h) \preceq (\tilde{Y}, \tilde{h})$$

if $Y \subset \tilde{Y}$ and $\tilde{h}$ is a linear extension of h.

We prove that any chain in P has an upper bound, and hence there exists a maximal element (Z, f) in P according to Zorn's lemma. The proof is then completed by demonstrating that Z = V.

The mapping p described in the previous theorem has two main cases.

Given a normed linear space X, the space of continuous linear functionals on X will be denoted by the symbol X*.

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1.1 Example Let W be a subspace of V and V be a normed linear space. Assume that g W is a continuous linear function. Then p(x) = gwx = gwx = gwx = gwx = gwx = gwx = gwx = gwx = (2.1). When we apply the above theorem to this situation, we get the following result.

1.2 Theorem (Hahn-Banach Theorem) Let R be a normed linear space, and V be a normed linear space over R. Allow W to be a V subspace and g: W R to be a continuous linear functional on W. Then there exists a continuous linear extension of g called f: V R.

$$\|f\|_{V^*} = \|g\|_{W^*}$$

This result is also true for normed linear spaces over C. If X is a normed linear space over the field of complex numbers, then let us write it in terms of its real and imaginary parts:

$$f = g + ih,$$

where i stands for a square root of −1. Then g and h are linear functionals, as long as we restrict ourselves to scalar multiplication by reals. Now, since
$f(ix) = if(x)$ for any $x \in X$, it follows easily that $h(x) = -g(ix)$. Thus, for any $x \in X$, we have,

$$f(x) = g(x) - ig(ix).$$

In other words, the real part of a linear functional over C is enough to de scribe the functional fully. If we restrict ourselves to scalar multiplication by reals only and consider X as a real normed linear space, then g is a contin uous linear functional and we can also show that $\|g\| = \|f\|_{X^*}$.

Thus, given a normed linear space V over C, a subspace W and a continuous linear functional g on W, we can write $g(x) = h(x) - ih(ix)$ for any $x \in X$, where h is a real valued linear functional over R. Then, by the previous theorem, we can find an extension $\tilde{h}$ of h and define

$$f(x) = \tilde{h}(x) - i\tilde{h}(ix).$$

Then it is easy to check that f is a norm preserving extension of g to all of V . Thus we have the following theorem.

Theorem 1.3 (Hahn-Banach Theorem) *Let V be a normed linear space over* C. *Let W be a subspace of V and let* $g : W \to$ C *be a continuous linear functional on W. Then there exists a continuous linear extension* $f : V \to$ C *of g such that*

$$kfk_{V^*} = kgk_{W^*}.$$

In more generic contexts, the geometric variants are also true. A topological vector space is one that has a Hausdorff topology and allows continuous operations like vector addition and scalar multiplication. If each point admits a neighbourhood system made up of convex sets, the space is said to be locally convex. For locally convex topolog ical vector spaces, the geometric versions of the Hahn-Banach theorems are true (cf. Rudin [5]).

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RICHNESS OF THE DUAL SPACE

One of the main consequences of the Hahn-banach theorem(s) is the fact that the dual of a normed linear space is well endowed with functionals and hence merits careful study.

Proposition 1.2 *Let V be a normed linear space and* $x_0 \in V$ *a non-zero vector. Then, there exists* $f \in V^*$ *such that* $\| f \| = 1$ *and* $f(x_0) = \|x_0\|$.

Proof: Let W be the one-dimensional space spanned by x_0. Define $g(\alpha x_0) = \alpha\|x_0\|$. Then $\|g\|_{W^*} = 1$. Hence, there exists $f \in V^*$ such that $\|f\|_{V^*} = 1$ and which extends g. Hence $f(x_0) = g(x_0) = \|x_0\|$.

If V is a normed linear space and if x and y are distinct points in V, then, clearly, there exists $f \in V^*$ such that $f(x) \neq f(y)$ (consider $x_0 = x - y \neq 0$). We say that V^* separates points of V.

Proposition 1.3 *Let W be a subspace of a normed linear space V . Assume that*

$\overline{W} \neq V$. *Then, there exists* $f \in V^*$ *such that* f 6≡ 0 *and such that* $f(x) \not\equiv 0$ *for all* $x \in W$.
Proof: Let $x_0 \in V \setminus \overline{W}$. Let $A = \overline{W}$ and $B = \{x_0\}$. Then A is closed, B is compact and they are non-empty and disjoint convex sets. Thus, there exists $f \in V^*$ and $\alpha \in$ R such that for all $x \in \overline{W}$,

$$\mathrm{Re}(f)(x) < \alpha < \mathrm{Re}(f)(x_0).$$

(We assume here that the base field is C; if it is R, then we can write f instead of Re(f).) Since W is a linear subspace, it follows that for all $\lambda \in$ R, we have $\lambda f(x) < \alpha$ for all $x \in W$. Now, since $0 \in W$, we have $\alpha > 0$. On the other hand, setting $\lambda = n$, we get that, for any $x \in W$,

$$\mathrm{Re}(f)(x) < \frac{\alpha}{n}$$

whence we see that $\mathrm{Re}(f)(x) \leq 0$ for all $x \in W$. Again, if $x \in W$, we also have $-x \in W$ and so $\mathrm{Re}(f)(-x) \leq 0$ as well and so $\mathrm{Re}(f)(x) = 0$ for all $x \in W$ and $\mathrm{Re}f(x_0) > \alpha > 0$. As already observed, the real part of a func tional determines the functional and so the proof is complete.

The preceding statement provides us with a very powerful tool for calculating the density of subspaces in a normed linear space. Let W be a subspace of V, which is a normed linear space. Assume that on V, we have a continuous linear functional that disappears on W. If we can prove that it disappears on all of V, then W = V, or, in other words, W is dense in V, follows from the preceding proposition. This is a phrase that is commonly used in a variety of contexts.

Proposition 1.4 *Let V be a normed linear space. Let* $x \in V$. *Then*

$$\|x\| = \sup_{f \in V^*,\ \|f\| \leq 1} |f(x)| = \max_{f \in V^*,\ \|f\| \leq 1} |f(x)| \qquad (1.4)$$

Proof: Clearly, $|f(x)| \leq \|f\|\,\|x\| \leq \|x\|$ when $\|f\| \leq 1$. On the other hand, by Proposition 1.4, there exists $f \in V^*$ such that $kfk = 1$ and $f(x) = \|x\|$ when x is non-zero. Thus the result is established for non-zero vectors and is trivially true for the null vector.

Compare the relation

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$$\|f\| = \sup_{x\in V,\ \|x\|\leq 1} |f(x)|, \tag{1.5}$$

which is a *definition*, with the relation (1.4), which is a *theorem*. In the former, the supremum need not be attained, while in the latter the supremum is always attained and hence is a maximum.

REFLEXIVE SPACES

The relation (1.4) is the starting point for the investigation of a very nice property of Banach spaces called reflexivity.

Let $x \in V$ and define

$$J_x(f) = f(x)$$

for $f \in V^*$. Then, by virtue of (1.4), it follows that $J_x \in (V^*)^* = V^{**}$ and that, in fact,

$$\|J_x\|_{V^{**}} = \|x\|_V .$$

Thus $J : V \to V^{**}$ given by $x \mapsto J_x$ is a norm preserving linear transforma tion. Such a map is called an isometry. The map J is clearly injective and maps V isometrically onto a subspace of V^{**}.

A Banach space V is said to be reflexive *if the canonical imbedding* $J : V \to V^{**}$*, given above, is surjective.*

Thus, if V is reflexive, we can identify the spaces V and V^{**}, using the isometry, J. Since V^{**}, being a dual space, is always complete, the notion of reflexivity makes sense only for Banach spaces. By applying Proposition 1.4 to V^*, it is readily seen that the supremum in (1.3) is attained for reflexive Banach spaces. A deep result due to R. C. James is that the converse is also true: *if V is a Banach space such that the supremum is attained in (1.3) for all* $f \in V^*$*, then V is reflexive*.

We saw above that the map J, being an isometry, is injective. If V is finite dimensional, then

$$\dim(V) = \dim(V^*) = \dim(V^{**})$$

and so J is surjective as well. Thus every finite dimensional space is reflexive.

Examples of infinite dimensional reflexive spaces are the sequence spaces

$$\ell_p = \left\{ x = (x_i) \mid \sum_{i=1}^{\infty} |x_i|^p < +\infty \right\}$$

with the norm

$$\|x\|_p = \left(\sum_{i=1}^{\infty} |x_i|^p\right)^{\frac{1}{p}},$$

where $1 < p < \infty$. The space ℓ_1 is *not* reflexive. Nor is the space of bounded sequences, ℓ_∞, with the norm

$$\|x\|_\infty = \sup_{1\leq i<\infty} |x_i|.$$

The closed subspaces *c* of convergent sequences and c_0 of sequences con verging to zero, of $`_\infty$ are not reflexive either. The space $C[0, 1]$ is also not reflexive.

One of the nice consequences of the Riesz representation theorem is that every Hilbert space is reflexive.

VECTOR VALUED INTEGRATION

Let us consider the unit interval [0, 1] endowed with the Lebesgue measure. Let *V* be a normed linear space over R. Let $\phi : [0, 1] \to V$ be a continuous mapping. We would like to give a meaning to the integral

$$\int_0^1 \varphi(t)\ dt$$

as a vector in *V* in a manner that the familier properties of integrals are preserved.

Using our experience with the integral of a continuous real valued func tion, one could introduce a partition

$$0 = x_0 < x_1 < \cdots < x_n = 1$$

and form Riemann sums of the form

$$\sum_{i=1}^{n} (x_i - x_{i-1})\varphi(\xi_i)$$

where $\xi_i \in [x_{i-1}, x_i]$ for $1 \leq i \leq n$, and define the integral (if it exists) as a suitable limit of such sums. Assume that such a limit exists and denote it by $y \in V$. Let $f \in V^*$. Then, by the continuity and linearity of *f*, it will follow that $f(y)$ will be the limit of the Riemann sums of the form

$$\sum_{i=1}^{n} (x_i - x_{i-1}) f(\varphi(\xi_i))$$

But since $f \circ$: [0, 1] → R is continuous, the above limit of Riemann sums is none other than

$$\int_0^1 f(\varphi(t))\ dt$$

Thus the integral of φ must satisfy the relation

$$f\left(\int_0^1 \varphi(t)\ dt\right) = \int_0^1 f(\varphi(t))\ dt$$

for all $f \in V^*$.

Notice that since V^* separates points of V, such a vector, if it exists, must be unique.

We use this to define the integral of a vector valued function. Let X be a set and let S be a σ-algebra of subsets of X, on which we have a meaure μ. We say that (X, S, μ) is a measure space. A mapping $\phi : X \to V$ is said to be *weakly measurable* if $f \circ \phi : X \to$ R (or C, if the space V is a complex normed linear space) is measurable for every $f \in V^*$.

Let (X, S, μ) be a measure space. Let V be a real normed linear space and let $: X \to$ R be a weakly measurable mapping. The integral of φ over X, denoted

$$\int_X \varphi(x)\ d\mu(x),$$

is that vector $y \in V$ which satisfies

$$f(y) = \int_X f(\varphi(x))\ d\mu(x)$$

for all $f \in V^*$.

Let

$$K = \{v \in V \mid \varphi_i(v) = 0,\ 1 \le i \le m\}.$$

The set K is not open in general and so we cannot apply the previous result and so we cannot deduce that $J'(u) = 0$.

Let us assume that J attains a relative extremum at a point $u \in K$ and that the gradient vectors $\{\varphi'(u)\}^m_{i=1}$ are all linearly independent. Then, by an application of the implicit function theorem, one can show that if $v \in V$ such that $\varphi'_i(u)(v) = 0$ for all $1 \le i \le m$, then $J'(u)(v) = 0$ as well (cf. Kesavan [3]).

Proposition 1.7 *Let V be a vector space and let $\{f_i\}^k_{i=0}$ be linear functionals on V such that*

$$\cap_{i=1}^k Ker(f_i) \subset Ker(f_0)$$

Then, there exist scalars $\{\lambda_i\}^k_{i=1}$ such that

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$$f_0 = \sum_{i=1}^{k} \lambda_i f_i.$$

CONVEX PROGRAMMING

In the previous section, we considered the relative extrema of functionals in the presence of constarits of th form $\varphi_i(x) = 0$, $1 \leq i \leq m$. We now consider the relative extrema of functionals under inequality constraints, *i.e.* constraints of the form $\varphi(x) \leq 0$, $1 \leq i \leq m$. The key step to this study is the analogue of Proposition, which is called the Farkas-Minkowski lemma.

A cone *in a real vector space V is a set C such that:*

(i) $0 \in C$;
(ii) *if* $x \in C$ *and* $\lambda \geq 0$, *then* $\lambda x \in C$.

Let v_i, $1 \leq i \leq n$ *be elements in a normed linear space V .*

Define

$$C = \left\{ \sum_{i=1}^{n} \lambda_i v_i \mid \lambda_i \geq 0,\ 1 \leq i \leq n \right\}$$

Then C is a closed convex cone.

It is easy to see that C is a convex cone. It is also not difficult to see that it is closed when the vectors are all linearly independent. It is possible to reduce the general case to the linearly independent case to get a complete proof of this lemma.

Proposition 1.7 (Farkas-Minkowski Lemma) *Let V be a real reflexive Banach space and let* $\{f_0, f_1, \cdots, f_n\}$ *be elements of* V^* *such that if for some* $x \in V$ *we have* $f_i(x) \geq 0$ *for all* $1 \leq i \leq n$, *then* $f_0(x) \geq 0$ *as well. Then, there exists scalars* $\lambda_i \geq 0$, $1 \leq i \leq n$ *such that*

$$f_0 = \sum_{i=1}^{n} \lambda_i f_i.$$

Separation of convex subsets of a real t.v.s.

Let X be a closed ane hyperplane of X t.v.s. over the field of real numbers. If A is contained in one of the two closed half-spaces specified by H and B is contained in the other, we say that H separates two disjoint subsets A and B of X. This attribute can be expressed in terms of functionals. Indeed, we may write that A and B are separated by H if and only if: H = L1(a) for some L : X R linear not identically zero and some a 2 R.

$$\exists\, a \in \mathrm{R} \text{ s.t. } L(A) \geq a \text{ and } L(B) \leq a.$$

where for any $S \subseteq X$ the notation $L(S) \leq a$ simply means $\forall s\ 2\ S$, $L(s) \leq a$ (and analogously for $\geq, <, >, =, \neq$).

If at least one of the two inequalities is strict, we say A and B are strictly separated by H. (Note that there are numerous definitions for rigorous separation in the literature; nonetheless, we will use the one given above.) In this subsection, we'll see if two disjoint convex subsets of a real t.v.s. can be separated, or at least strictly separated.

2.1. Proposition Let X be a t.v.s. over real numbers, with A and B being two disjoint convex subsets of X.

a) If A is open and B is nonempty, then there exists a closed ane hyperplane H of X separating A and B, i.e. a R and a functional L: X R linear not identically zero s.t. L(A) a and L(B) an are two different types of L(A) a.

b) If B is also open, the hyperplane H can be chosen to strictly separate A and B, i.e. a R and L: X R linear not identically zero s.t. L(A) a and L(B) an are two different types of L(A) a.

c) If A is a cone and B is open, then a can be zero, implying that L: X R linear is not identically zero s.t. L(A) 0 and L(B) 0 are both zero.

We call the basic closed semialgebraic set created by S the following given a finite collection of polynomials S:= g1,...,gs.

$K_S := \{\underline{x} \in \mathrm{R}^d : g_i(\underline{x})\ 0, i = 1,\ldots,s\}$.

Definition 2.2.2. *A subset M of* R[$\underline{x}$] *is said to be a* quadratic module *if* $1 \in M$, $M + M \subseteq M$ *and* $h^2 M \subseteq M$ *for any* $h \in \mathrm{R}[\underline{x}]$.

Note that each quadratic module is a cone in R[$\underline{x}$].

Definition 2.2.3. *A quadratic module M of* R[$\underline{x}$] *is called* Archimedean *if there exists* $N \in \mathrm{N}$ *s.t.* $N-(\sum_{i=1}^{d} x^2{}_i) \in M$.

For $S := \{g_1,\ldots,g_s\}$ finite subset of R[$\underline{x}$], we define the *quadratic module generated by S* to be:

$$M_S := \left\{ \sum_{i=0}^{s} \sigma_i g_i : \sigma_i \in \sum \mathbb{R}[\underline{x}]^2, i = 0, 1, \ldots, s \right\}$$

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where $g_0 := 1$.

Note that $M_S \to Psd(K_S)$ and M_S is the smallest quadratic module of R[x] *containing S.*

Consider now the finite topology on R[x] which we have proved to be the finest locally convex topology on this space and which we therefore denote by φ. we get that

$$\overline{M_S}^{\varphi} = (M_S)^{\vee\vee}_{\varphi} \tag{2.3}$$

Moreover, the *Putinar Positivstellesatz* (1993), a milestone result in real al gebraic geometry, provides that if M_S is Archimedean then

$$Psd(K_S) \subseteq \overline{M_S}^{\varphi}. \tag{2.4}$$

Note that M_S is Archimedean implies that K_S is compact while the converse is in general not true (see e.g. M. Marshall, Positive polynomials and sum of squares, 2008).

Combining (2.3) and (2.4), we get the following result.

Let $S := \{g_1,...,g_s\}$ be a finite subset of R[x] *and L :* R[x] $\to$ R *linear. Assume that M_S is Archimedean. Then there exists a K_S-representing measure μ for L if and only if $L(M_S) \geq 0$, i.e. $L(h^2 g_i) \geq 0$ for all $h \in$* R[x] *and for all $i \in \{1,...,s\}$.*

EQUIVALENT DUAL NORMS

The following characterization of equivalent dual norms can be found in [3, Fact 5.4]:

Fact 2.1 (Deville, Godefroy, and Zizler, 1993). *Let* X *be a real Banach space. Let* $|\cdot|$ *be an equivalent norm on* X^*. *The following conditions are equiva lent:*

(1) *The norm* $|\cdot|$ *is a dual norm.*

(2) *The Banach-Aloaglu Theorem is verified, in other words,* $B_{|\cdot|}$ *is* w^**-closed.*

(3) *The norm* $|\cdot| : (X^*, w^*) \to$ R *is lower semi-continuous.*

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Our contribution to this topic consists of showing that an equivalent norm on a dual space is a dual norm if and only if the Goldstine Theorem is verified. This will be bring interesting consequences on the extension of dual norms.

Let X be a real Banach space. Let $|\cdot|$ be an equivalent norm on X^*. Consider the equivalent norm on X given by $\lfloor\cdot\rfloor := |\cdot|^*|_X$. We have the following basic facts:

- $B_{\lfloor\cdot\rfloor} = B_{|\cdot|^*} \cap X$.

- $|\cdot|$ is a dual norm on X if and only if $|\cdot| = \lfloor\cdot\rfloor^*$.

- $\lfloor\cdot\rfloor^{**}|_X = |\cdot|^*|_X$.

By taking into consideration we are capable of carrying out the desired characterizations.

Let X *be a real Banach space. Let* $|\cdot|$ *be an equivalent norm on* X^*. *The following conditions are equivalent:*

(1) *The norm* $|\cdot|$ *is a dual norm.*

(2) *The Goldstine Theorem is verified, in other words,*

$$\mathrm{cl}_{w^*}\left(\mathsf{B}_{|\cdot|^*} \cap X\right) = \mathsf{B}_{|\cdot|^*}.$$

Proof. Let $\lfloor\cdot\rfloor := |\cdot|^*|_X$. For every $x^* \in X^*$ we have that

$$\begin{aligned}\lfloor x^*\rfloor^* &= \sup\left\{x^*(x) : x \in \mathsf{B}_{\lfloor\cdot\rfloor}\right\} \\ &= \sup\left\{x(x^*) : x \in \mathsf{B}_{\lfloor\cdot\rfloor}\right\} \\ &= \sup\left\{x(x^*) : x \in \mathsf{B}_{|\cdot|^*} \cap X\right\} \\ &= \sup\left\{x^{**}(x^*) : x^{**} \in \mathsf{B}_{|\cdot|^*}\right\} \quad \text{(Goldstine Theorem)} \\ &= |x^*|.\end{aligned}$$

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The following characterization of equivalent dual norms solves an open problem on the extension of equivalent norms.

Let X *be a real Banach space. Let* $|\cdot|$ *be an equivalent norm on* X^*. *Consider the equivalent norm on* X *given by* $\lfloor\cdot\rfloor := |\cdot|^*|_X$. *The following conditions are equivalent:*

(*1*) *The norm* $|\cdot|$ *is a dual norm.*

(*2*) $\lfloor\cdot\rfloor^{**} = |\cdot|^*$.

Proof. Assume that $\lfloor\cdot\rfloor^{**} = |\cdot|^*$. We will show that the Goldstine Theorem is verified by $|\cdot|^*$, that is,

$$\mathrm{cl}_{w^*}\left(\mathsf{B}_{|\cdot|^*} \cap X\right) = \mathsf{B}_{|\cdot|^*}.$$

Indeed, by bearing in mind Remark 2.2 we have that

$$\mathrm{cl}_{w^*}\left(\mathsf{B}_{|\cdot|^*} \cap X\right) = \mathrm{cl}_{w^*}\left(\mathsf{B}_{\lfloor\cdot\rfloor}\right) = \mathsf{B}_{\lfloor\cdot\rfloor^{**}} = \mathsf{B}_{|\cdot|^*}$$

in virtue of the fact that the bidual norm $\lfloor\cdot\rfloor^{**}$ verifies the Goldstine Theorem. Now it only suffices to apply the previous theorem.

An informally open question among Functional Analysts is to determine whether two equivalent dual norms on a dual space are equal provided that they coincide on a w*-dense subspace. The next example shows that this is not true in virtue.

Let X be a non-reflexive real Banach space. Let $|\cdot|$ be an equiv alent norm on X^* which is not a dual norm. Consider the equivalent norm on X given by $\lfloor\cdot\rfloor := |\cdot|^*|_X$. In accordance to we have that both $\lfloor\cdot\rfloor^{**}$ and $|\cdot|^*$are equivalent dual norms on X^{**} which coincide on the w*-dense subspace X but which are not equal.

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CONSEQUENCES OF HAHN BANACH THEOREM

First application, we see that the dual space $X' \neq \{0\}$ if $X \neq \{0\}$.

Corollary 3.1. *Let X be a normed linear space and* $x_0 \in X$ *a non-zero vector. Then there are non-trivial continuous linear functionals on X. In particular, there exists a* $f \in X'$ *such that* $||f|| = 1$ *and* $f(x_0) = ||x_0||$.

Proof. Define $M = \{\alpha x_0 : \alpha \in K\}$. Define $f : M \to K$ by $f(\alpha x_0) = \alpha ||x_0||$. It is clear that f is continuous linear functional on M. Then by Hahn-Banach theorem, there exists $\bar{f} \in X'$ such that $\tilde{f}|M = f$ on M and $||\tilde{f}|| = ||f|| = 1$.

From the above corollary, we get the following results.

Corollary 3.2. *Let X be a normed linear space. Then*

$$||x|| = \sup_{0 \neq f \in X'} \frac{|f(x)|}{||f||}.$$

Proof. For $0 \neq f \in X'$, we get

$$|f(x)| \leq ||f|| \, ||x|| \implies ||x|| \geq \frac{|f(x)|}{||f||}$$
$$\implies ||x|| \geq \sup_{0 \neq f \in X'} \frac{|f(x)|}{||f||}.$$

If $x = 0$, then the result is trivial. If x 6= 0, then the result follows by corollary 3.1.

Corollary 3.3. *Let X be a normed linear space. If* $f(x) = 0$ *for all* $f \in X'$, *then* $x = 0$.

Corollary 3.4. *Let Y be a proper closed subspace of a normed space X. Let* $x_0 \in X \setminus Y$ *be arbitrary and*

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$$\delta = \inf_{y \in Y} ||x_0 - y|| = \ dist(x_0, Y).$$

Then there exists a continuous linear functional $\widetilde{f}$ *on* X' *such that*

$$\widetilde{f}|_Y = 0, \widetilde{f}(x_0) = \delta, \ \ and \ ||\widetilde{f}|| = 1.$$

Proof. The proof is similar to corollary 3.1. Consider the subspace $M = \{y+\alpha x_0 : y \in Y, \ \alpha \in \mathrm{K}\}$. Define $f : M \to \mathrm{K}$ by

$$f(y + \alpha x_0) = \alpha\delta.$$

Notice that the number $\delta > 0$ (why?), hence $f \neq 0$. Now consider

Using the Hahn-Banach theorem, the result follows (verify).

$$\begin{aligned} |f(y+\alpha x_0)| &\leq |\alpha|\delta = |\alpha| \inf_{y \in Y} ||x_0 - y||, \ \forall \alpha \in \mathbb{K} \\ &\leq |\alpha|||x_0 - y||, \ \forall y \in Y \\ &\leq ||\alpha x_0 - \alpha y||, \ \forall y \in Y \\ &\leq ||\alpha x_0 + y||, \ \forall y \in Y, \alpha \in \mathbb{K}. \quad \text{(Why?)} \end{aligned}$$

Corollary 3.5. *Let* $T : H \to H$ *be a bounded linear operator on a Hilbert space H. If* $T = T^*$, *then* $||T|| = \sup\{|\langle Tx, x\rangle| : ||x|| = 1\}$.

Proof. Since the set $\{|\langle Tx, x\rangle| : ||x|| = 1\}$ is bounded by $||T||$ (why?), the supremum $\alpha = \{|\langle Tx, x\rangle| : ||x|| = 1\}$ exists and $\alpha \leq ||T||$.

Let $x, y \in H$, and $||x|| = ||y|| = 1$. Consider

$$\langle T(x \pm y), x \pm y)\rangle = \langle Tx, x\rangle \pm \langle Tx, y\rangle \pm \langle Ty, x\rangle + \langle Ty, y\rangle$$

$$= \langle Tx, x\rangle \pm \langle Tx, y\rangle \pm \langle y, T^*x\rangle + \langle Ty, y\rangle$$

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$$= \langle T x, x\rangle \pm 2 \text{ Re } \langle T x, y\rangle + \langle T y, y\rangle. \ (\because T = T^*)$$

Thus

$$\langle T(x + y), x + y)\rangle - \langle T(x - y), x - y)\rangle = 4 \text{ Re } \langle T x, y\rangle.$$

From this, we get

$$4 \text{ Re } \langle T x, y\rangle \leq \alpha(||x + y||^2 + ||x - y||^2) \ (\because \langle T z, z\rangle \leq \alpha||z||^2)$$

$$\leq 2\alpha(||x||^2 + ||y||^2)$$

$$\leq 4\alpha \ (\forall ||x|| = ||y|| = 1). \qquad (3.1)$$

Since $| \langle T x, y\rangle | = e^{i\theta} \langle T x, y\rangle = \langle T x, e^{-i\theta} y \rangle$ and $||e^{-i\theta}y|| = 1$, we get

$$| \langle T x, y\rangle | \leq \alpha, \ \forall ||x|| = ||y|| = 1 \text{ [Using (3.1)]}$$

$$\sup_{||y|| = 1} | \langle T x, y\rangle | \leq \alpha,$$

$$||T x|| \leq \alpha, \ \forall ||x|| = 1 \text{ [Using corollary 3.2]}$$

$$\Rightarrow ||T|| \leq \alpha.$$

Conclusion:

After discussing the importance, relevance, and other aspects of Hahn Banach theorems, we now discuss some of their limits, as well as the limitations of methods based on these theorems. In fact, many theorems and methods of Functional Analysis share these restrictions. All of these Hahn Banach theorems' claims are existence statements, meaning they assert the existence of some linear functionals. All of the proofs, however, are nonconstructive in character. This means that, even when the vector space under examination is finite dimensional, these proofs provide no guidance on how to find the linear functional whose existence is proved by a theorem. (This is comparable to several proofs of the Fundamental Theorem of Algebra,

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none of which explain how to locate the root of a given polynomial, despite the fact that the Theorem states that every such polynomial must have a root.) Consider the case where we are given two finite sets in R 10. One technique to provide these sets is to provide two matrices, each with ten rows. Let's call these two sets' convex hulls A and B. (Remember that a set's convex hull is the smallest convex set that contains the given set.) Let's have a look at the following scenario: To evaluate whether A and B are disjoint, and if so, to locate a hyperplane in R 10 that separates A and B. At first glance, the Hahn Banach Separation Theorem appears to be a viable solution to this problem. It is useful in the sense that if A and B are disjoint, the Theorem states that a hyperplane in R 10 exists between them. However, it makes no mention of how to locate such a hyperplane. To address this issue, a variety of approaches must be applied..)

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