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Rudin-Keisler Order on Tensor Product of Ultrafilters

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ABSTRACT

Tensor product is a useful tool to identify an ultrafilter of the Cartesian product of two discrete semigroups which was first introduced by S Kochen in his paper [10].

In this paper we show that binary compositions of any two ultrafilters in the Stone - Čech compactification of a discrete semigroup are bounded above with respect to Rudin-Keisler ordering by tensor product of them. Also we have established that Rudin-Keisler ordering is preserved under some special type of homomorphisms on the Stone - Čech compactification βS.

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Key Words: Tensor product, ultrafilter, Rudin-Keisler ordering, Stone - Čech compactification.

1 INTRODUCTION

Given a nonempty set X, a family $\mathfrak{F}$ of subsets of X will be a filter on X if (i) $\emptyset \notin \mathfrak{F}$, (ii) $A \cap B \in \mathfrak{F}$
when $A, B \in \mathfrak{F}$ and (iii) if $A \in \mathfrak{F}$ and $A \subseteq B$ then $B \in \mathfrak{F}$.

A filter $\mathfrak{U}$ is called an ultrafilter on X if $\mathfrak{U}$ is not properly contained in any filter. An ultrafilter $\mathfrak{U}$ which contains a singletone set $\{x_0\}$ as a member is called a principal ultrafilter. We denote this ultrafilter by x_0. For a principal ultrafilter, $\bigcap_{U \in \mathfrak{U}} U$ = a singletone set. For further studies on ultrafilters, we refer [7]. A compactification of a space X is a compact Hausdorff space Y such that there is a topological embedding $e : X \to Y$ with e(X) dense in Y. Whereas the Stone - Čech compactification of a Tychonoff space X is a compactification βX having the property that: if $f: X \to Y$ is a continuous function on any compact Hausdorff space Y then there is a unique continuous function $\tilde{f}: \beta X \to Y$ with $\tilde{f}|_X = f$.

The Stone - Čech compactification of a Tychonoff space X taking maximal Z-ideals of X as points has been studied in [8] which is equivalent for a discrete space X by taking all its ultrafilters. For a discrete semigroup $(S, \cdot)$ its Stone - Čech compactification βS consisting of all ultrafilters of S is topologized by taking the collection $\{\widehat{A} : A \subseteq S\}$ as a base for the topology on βS, where $\widehat{A} = \{p \in \beta S : A \in p\}$. The above mentioned base is also a base for the closed sets in βS. Thus the Stone - Čech compactification _S of a discrete semigroup S is a zero- dimensional

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space. We can identify each element $s \in S$ with a principal ultrafilter, so that $S \subseteq \beta S$. Then the semigroup operation $\cdot$ on S can be extended to a binary operation on βS as follows:

for $s \in S, q \in \beta S$, $s \cdot q = \widetilde{\lambda_s}(q)$, where $\widetilde{\lambda_s} : \beta S \to \beta S$ is the continuous extension of $\lambda_s : S \to S (\subseteq \beta S)$ defined by $\lambda_s(x) = s \cdot x$.

Now for $p, q \in \beta S$, $p \cdot q = \widetilde{\rho_q}(p)$, where $\widetilde{\rho_q} : \beta S \to \beta S$ is the continuous extension of $\rho_q : S \to \beta S$ defined by $\rho_q(x) = x \cdot q$ and $(\beta S, \cdot)$ is also a semigroup.

Also it is a compact right topological (follows form [4]) semigroup. From Ellis theorem ([4], theorem 2.5), it is clear that $(S, \cdot)$contains an idempotent element.

2 HOMOMORPHISMS ON INVERSE SEMIGROUP

Definition 2.1. Suppose $(S, \cdot)$ is a discrete inverse semigroup, $A \subseteq S$ and $p \in \beta S$.We define $A^{-1} = \{x' : x \in A\}$, and $p' = \{A^{-1} \subseteq S : A \in p\}$.

Obviously $(A^{-1})^{-1} = A$ and $(p')' = p$.

Proposition 2.2. $p \in \beta S$ if and only if $p' \in \beta S$.

Proof. Suppose $p \in \beta S$. Clearly then $\emptyset \notin p'$. Now if $A, B \in p'$ then $A^{-1}, B^{-1} \in p$. p being an ultrafilter, $A^{-1} \cap B^{-1} \in p$. Then $A \cap B = (A^{-1} \cap B^{-1})^{-1} \in p'$. Again if $A \in p'$ and $A \subseteq B$ then

$A^{-1} \subseteq B^{-1}$ implies $B^{-1} \in p$ implies $B \in p'$. So p' is a filter on S. Suppose $\mathfrak{F}$ be a filter such that $p' \subseteq \mathfrak{F}$. Then $p = (p')' \subseteq \mathfrak{F}'$. Since p is an ultrafilter, $\mathfrak{F}' = p$ and hence $p' = \mathfrak{F}$. The converse follows from $(p')' = p$.

Notation: For a subset A of a semigroup $(S, \cdot)$ if $s \in S$ then

(a) $s^{-1}A = \{t \in S : s \cdot t \in A\}$ and

(b) $As^{-1} = \{t \in S : t \cdot s \in A\}$.

The following theorem follows from the continuity of $\widetilde{\lambda_s}$ and $\widetilde{\rho_q}$.

Theorem 2.3. [4] For a discrete semigroup $(S, \cdot)$ if $x \in S$, $p, q \in \beta S$ then

(a) $x \cdot p = \{A \subseteq S : x^{-1}A \in p\}$

(b) $p \cdot q = \{A \subseteq S : \{x \in S : x^{-1}A \in q\} \in p\}$.

Definition 2.4. Suppose (X, τ) be a topological space and S be a discrete space. For $p \in \beta S$, we say that $\lim_{s \to p}(x_s) = x$ if for any $x \in U \in \tau$, the set $\{s \in S : x_s \in U\} \in p$.

Clearly, for any $p \in \beta S$, $\lim_{s \to p}(s) = p$.

Theorem 2.5. If $\tilde{f}$ is the continuous extension of a map $f : S \to T$ (where S is a discrete semigroup

and T is a compact Hausdorff right topological semigroup), then $\tilde{f}(p) = \lim_{s \to p} f(s)$.

Theorem 2.6. For an inverse semigroup $(S, \cdot)$ the mapping $\gamma : \beta S \to \beta S$ defined by $\gamma(p) = p'$ is

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a homeomorphism.
Proof : Clearly γ is a bijective mapping. As $\gamma = (\gamma^{-1})^{-1}$, to prove the theorem it is enough to show
that γ is continuous. Suppose $\widehat{A}$ be a basic open set containing $\gamma(p) = p'$. Then $A \in p'$, implies $A^{-1} \in p$, implies $p \in \widehat{(A^{-1})}$. It can be easily verified that $\gamma(\widehat{(A^{-1})}) = (\gamma(A))^{-1}$. As every element of S
is a principal ultrafilter in βS, γ is the continuous extension of the map $x \rightarrow x'$ in S.

Theorem 2.7. For a commutative inverse semigroup $(S, \cdot)$ if $a \in S$, $p, q \in \beta S$ then
(a) $(a \cdot p)' = a' \cdot p'$
(b) $(p \cdot q)' = p' \cdot q'$.
(c) p is an idempotent if and only if p' is an idempotent.

Proof. (a) For any $a \in S$ and for any $x \in S$ we see that $\gamma((\lambda_a)(x)) = (\lambda_a(x))' = (a \cdot x)' = (x \cdot a)' = a' \cdot x' = \lambda_a'(\gamma(x))$. So $\gamma \circ \lambda_a$ and $\lambda_a' \circ \gamma$ are two continuous functions agreeing in S, a dense subspace of the Hausdorff space βS. Therefore $\gamma \circ \lambda_a = \lambda_a' \circ \gamma$ on βS. Thus $(a \cdot p)' = a' \cdot p'$.

(b) For any $q \in \beta S$ and for any $x \in S$ we see that $\left(\gamma \circ \rho_q\right)(x) = \left(\rho_q(x)\right)' = \gamma(x \cdot q) = (x \cdot q)' = x' \cdot q'\left(\text{from (a)}\right) = \rho_{q'}\left(\gamma(x)\right) = \rho_{q'} \circ \gamma(x)$. By similar arguments $\gamma \circ \rho_q = \rho_{q'} \circ \gamma$ on βS.
Thus $(p \cdot q)' = p' \cdot q'$.
(c) Follows directly from (b).

From the theorem 2.6 and 2.7 we arrive at the conclusion stated in the following corollary,

Corollary 2.8. The mapping γ is a topological isomorphism on βS when S is a commutative inverse semigroup.

3. Tensor Product and Rudin-Keisler order

Definition 3.1 Suppose G and G_1 are two discrete groups and $p \in \beta G$, $q \in \beta G_1$. The tensor product of p and q is defined by
$p \otimes q = \{U \subseteq G \times G_1 : \{a \in G: \{b \in G_1: (a, b) \in U\} \in q\} \in p\}$.

Clearly, $p \otimes q$ is an ultrafilter of $G \times G_1$. From the definition, it is clear that, a subset $U (\subseteq G \times G_1) \in p \otimes q$ if and only if U contains a set of the form $\{(a, b): a \in A \text{ and } b \in B_a\}$, where $A \in p$ and $B_a \in q$ for each $a \in A$. Also if $a \in \beta G$, $b \in \beta G_1$ are principal ultrafilters, then $a \otimes b$ is also the principal ultrafilter (a, b).

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If $\phi: G \times G \to G$ is any bijection, then we can consider $p \otimes q$ as the ultrafilter of G identified by $\tilde{\phi}(p \otimes q)$, where $\tilde{\phi}: \beta(G \times G) \to \beta G$ is the continuous extension of ϕ.Clearly for any $a, b \in G$, $a \otimes b$ will then be identified by the principal ultrafilter $\phi(a, b)$.

Theorem 3.2 [8] For any two discrete semigroups S and T, if $p \in \beta S$, $q \in \beta T$, then $p \otimes q = \lim_{s\to p} \lim_{t\to q} (s, t)$

Proposition 3.3 Suppose G and G_1 are two discrete groups and π_G, π_{G_1} are projection mappings from $G \times G_1 \to G$ and $G \times G_1 \to G_1$ respectively. For $a \in G, b \in G_1$, $p, r \in \beta G$, $q, s \in \beta G_1$,

(a) $\widetilde{\pi_G}(p \otimes q) = p$ and $\widetilde{\pi_{G_1}}(p \otimes q) = q$.

(b) $p \otimes q = r \otimes s$ if and only if $p = r$ and $q = s$.

(c) $a \otimes b + r \otimes s = (a + r) \otimes (b + s)$.

Proof. (a) For any $a \in G, b \in G_1$, $\widetilde{\pi_G}(p \otimes q) = \pi_G\left(\lim_{(a,b)\to(p,q)}(a, b)\right) = \lim_{(a,b)\to(p,q)} \pi_G (a, b) = \lim_{a\to p} \lim_{b\to q}(a) = p$. Similarly, we can show that $\widetilde{\pi_{G_1}}(p \otimes q) = q$.

(b) If $p = r$ and $q = s$, then obviously, $p \otimes q = r \otimes s$. Conversely, if $p \otimes q = r \otimes s$ then from (a), $p = \widetilde{\pi_G}(p \otimes q) = \widetilde{\pi_G}(r \otimes s) = r$ and $q = \widetilde{\pi_{G_1}}(p \otimes q) = \widetilde{\pi_{G_1}}(r \otimes s) = s$.

(c) From theorem 3.2, $a \otimes b + r \otimes s$

$= (a, b) + \lim_{c\to r} \lim_{d\to s} (c, d)$ (where $c \in G, d \in G_1$)

$= \lim_{c\to r} \lim_{d\to s}(a, b) + (c, d))$

$= \lim_{c\to r} \lim_{d\to s}(a + c, b + d)$

$= (a + r) \otimes (b + s)$.

Theorem 3.4 [9] Let $(S, +)$ and $(T, +)$ be semigroups, let $p, r \in \beta S$, and let $q, w \in \beta T$. If $u \in \beta S$, $v \in \beta T$, and $(p \otimes q) + (r \otimes w) = u \otimes v$, then $u = p + r$ and $v = q + w$.

Theorem 3.5 If $p \otimes q$ is an idempotent in $\beta(G \times G_1)$ then p and q are idempotents in βG and βG_1 respectively.

Proof $p \otimes q$ is an idempotent in $\beta G \times \beta G_1 \implies (p \otimes q) + (p \otimes q) = p \otimes q$. From theorem 3.4, this implies $p + p = p$ and $q + q = q$. Then p and q are idempotents in βG and βG_1 respectively.

Theorem 3.6 [9] Let $(S, +)$and $(T, +)$ be infinite semigroups and assume that $w, z \in \beta T$ and $Q = \{t \in T : t + w = z\}$ is infinite. Let $q \in Q^*$ and let $p, r \in \beta S$. Then $(p \otimes q) + (r \otimes w) = (p + r) \otimes (q + w)$.

From Theorem 3.4 and 3.6, we can say that

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Theorem 3.7 Let $(S,+)$and $(T,+)$ be infinite semigroups and for any $w, z \in \beta T$ the set $\{t \in T : t + w = z\}$ is infinite. Then for $\in S^*$, $q \in T^*$, $p \otimes q$ is an idempotent in $(S \times T)^*$ if and only if p and q are idempotents.

Remark 3.8 From proposition 3.3 (a) it is clear that an ultrafilter $p \otimes q$ of $G \times G_1$ is a principal ultrafilter if and only if both p and q are principal ultrafilters of G and G_1respectively.

Theorem 3.9 [8] Let S and T be arbitrary discrete spaces and let $f : S \to S \subseteq \beta S$ and $g : T \to T \subseteq \beta T$ be arbitrary maps. Define $h : S \times T \to S \times T \subseteq \beta(S \times T)$ by $h(s,t) = (f(s), g(t))$. Then for every $p \in \beta S$ and $q \in \beta T$, $\tilde{h}(p \otimes q) = \tilde{f}(p) \otimes \tilde{g}(q)$. Furthermore, $\tilde{h}$ is injective if f and g are injective, and $\tilde{h}$ is surjective if f and g are surjective. If S and T are semigroups and f and g are homomorphisms, then $\tilde{h}$ is also a homomorphism.

Theorem 3.10 For each $a \in S$, the mapping $\lambda_a : \beta T \to \beta(S \times T)$ defined by $\lambda_a(q) = a \otimes q$ is an open continuous homomorphism.

Proof : For any $q \in \beta T$, suppose $\widehat{L}$ be a basic open set containing $\lambda_a(q) = a \otimes q$. Then $L \in a \otimes q$, which implies L contains a set of the form $\{(s,t): s \in A \text{ and } t \in B_s\}$, where $A \in a$ and $B_s \in q$ for each $s \in A$. . Suppose $B = B_a$. Then $\widehat{B}$ is an open set in βTcontaining q and $\forall\, r \in \widehat{B}$, $B \in r$ which implies $L \in a \otimes r$, ie $a \otimes r \in \widehat{L}$. Thus $\lambda_a(\widehat{B}) \subseteq \widehat{L}$ which implies that λ_a is a continuous homomorphism.

To prove that λ_a is an open mapping, suppose $\widehat{B}$ be a basic open set in βT. Then $B \subseteq T$. Now if $l \in \lambda_a(\widehat{B})$
then $l = a \otimes q$, for some $q \in \widehat{B}$. Clearly then $l \in \widehat{\{a\} \times B} \subseteq \lambda_a(\widehat{B})$: indeed, $\{a\} \times B \in m$ implies for every $U(\subseteq S \times T) \in m$, $B \cap \{t \in T: (a,t) \in U\} \neq \square$. Therefore, $\{t \in T: (a,t) \in U\} \in n$ for some $n \in \widehat{B}$. This implies
$\{s \in S: \{t \in T: (s,t) \in U\} \in n\} \in a$, which implies $U \in a \otimes n$ and so $m = a \otimes n$. Thus $\widehat{\{a\} \times B} \subseteq \lambda_a(\widehat{B})$. Consequently, λ_a is an open mapping.

The Rudin-Keisler order on the Stone - Čech compactification of a discrete semigroup is an important tool in analyzing spaces of ultrafilters. It shows how one ultrafilter can be essentially different from another.

Definition 3.11 [4] Let $(S, \cdot)$ be a discrete semigroup. For $p, q \in \beta S$, the Rudin-Keisler order $\leq_{RK}$
is defined by $p \leq_{RK} q$ if there is a function $f : S \to S$ for which $\tilde{f}(q) = p$.

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For $p, q \in \beta S$, if $p \leq_{RK} q$ and $q \leq_{RK} p$ then we write $p \approx_{RK} q$. Also if $p \leq_{RK} q$ and $q \nleq_{RK} p$ then we write $p <_{RK} q$.

Theorem 3.12 Let S be a discrete semigroup and $p \in \beta S$. Then $p \leq_{RK} q$ for every $q \in \beta S$ if and only if $p \in S$.

Proof. Suppose $p \leq_{RK} q$ for every $q \in \beta S$, then for each $q \in \beta S$, there is a function $f_q : S \to S$ such that $\widetilde{f_q}(q) = p$. Since $\widetilde{f_q}: \beta S \to \beta S$ is the continuous extension of f_q, for any $q \in S$, $\widetilde{f_q}(q) = f_q(q) \in S$ implies that $p \in S$. Conversely, if $p \in S$, then the mapping $f: S \to S$ defined by $f(s) = p \ \forall s \in S$ is a continuous mapping and its continuous extension $\tilde{f}: \beta S \to \beta S$ (being unique) must be the constant function. Therefore $\tilde{f}(q) = p \ \forall q \in \beta S$, which implies that $p \leq_{RK} q$ for every $q \in \beta S$.

For further study, we use the following result from [4].

Theorem 3.13 For a discrete space S and $p, q \in \beta S$ the following statements are equivalent.

(a) $p \approx_{RK} q$.

(b) $p \leq_{RK} q$ and if $f: S \to S$ with $\tilde{f}(q) = p$, there exists some $Q \in q$ such that $f_{|Q}$ is injective.

(c) There exists $f: S \to S$ and $Q \in q$ such that $\tilde{f}(q) = p$ and $f_{|Q}$ is injective.

(d) There is a bijection $g: S \to S$ such that $\tilde{g}(q) = p$.

For any bijective mapping $\phi: G \times G \to G$, and for any $p, q \in \beta G$ as we can consider $p \otimes q$ as an element of βG, $\otimes$ can be treated as a binary operation on βG. However in the following theorem we see that all the tensor product of $p, q \in \beta G$ irrespective of the bijective mappings $\phi: G \times G \to G$, are Rudin-Keisler equivalent.

Theorem 3.14 Suppose $\phi: G \times G \to G$ and $\psi: G \times G \to G$ be two bijective mappings and $p, q \in \beta G$. Then $p \otimes_\phi q \approx_{RK} p \otimes_\psi q$, where $\otimes_\phi$ and $\otimes_\psi$ are the binary operations by the tensor product under the maps ϕ and ψ respectively.

Proof: Here $p \otimes_\phi q = \tilde{\phi}(p \otimes q) = \lim_{a \to p} \lim_{b \to q} \phi(a, b)$ and $p \otimes_\psi q = \tilde{\psi}(p \otimes q) = \lim_{a \to p} \lim_{b \to q} \psi(a, b)$. Suppose $f = \psi \circ (\phi)^{-1}$. Then $f: G \to G$ is a bijective mapping. Now $\tilde{f}(p \otimes_\phi q) = \tilde{f}\left(\lim_{a \to p} \lim_{b \to q} \phi(a, b)\right) = \lim_{a \to p} \lim_{b \to q} f(\phi(a, b)) = \lim_{a \to p} \lim_{b \to q} \psi(a, b) = \tilde{\psi}(p \otimes q) = p \otimes_\psi q$. Therefore from theorem 3.13, $p \otimes_\phi q \approx_{RK} p \otimes_\psi q$.

From the proposition 3.3 (a) we see that if $\pi_1: \begin{matrix} G \times G \to G \\ (x, y) \to x \end{matrix}$ and $\pi_2: \begin{matrix} G \times G \to G \\ (x, y) \to y \end{matrix}$ be two projection mappings then $\widetilde{\pi_1}(p \otimes q) = p$ and $\widetilde{\pi_2}(p \otimes q) = q$. Therefore:

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Theorem 3.15 For any discrete space G, if $p, q \in \beta G$, then $p \leq_{RK} p \otimes q$ and $q \leq_{RK} p \otimes q$.

From theorem 3.13, theorem 3.14 and 3.15 we get the following result:

Corollary 3.16 For any discrete space G, if $p, q \in G^*$, then $p <_{RK} p \otimes q$ and $q <_{RK} p \otimes q$.

From the following theorem [4], we shall show that $p \otimes q$ is an upper bound of the set of all binary compositions of p and q.

Theorem 3.17[4] Suppose G be a discrete space and $\circ$ be a binary operation on G. For $a, b, p, q \in \beta G$, if $a \leq_{RK} p$, $b \leq_{RK} q$, then $a \circ b \leq_{RK} p \otimes q$.

Corollary 3.18 Suppose G be a discrete space and $\circ$ be a binary operation on G. For $p, q \in \beta G$, $p \circ q \leq_{RK} p \otimes q$.

Proof: The proof follows from theorem 3.17 by taking a = p and b= q.

Corollary 3.19 Suppose G be a discrete space and $p, q, r, s \in \beta G$. If $p \leq_{RK} r$, $q \leq_{RK} s$, then $p \otimes q \leq_{RK} r \otimes s$.

Proof: The proof follows from theorem 3.17 by considering the binary operation $\circ$ as the tensor product $\otimes$.

Corollary 3.20 Suppose G be a discrete space and $p, q, r, s \in \beta G$. If $p \approx_{RK} r$, $q \approx_{RK} s$, then $p \otimes q \approx_{RK} r \otimes s$.

Proof: The proof follows from corollary 3.19.

4. Homomorphism on □□ preserving Rudin-Keisler order

In this section we discuss about the continuous homomorphisms on the Stone - Čech compactification βS which preserve the Rudin-Keisler oreder of ultrafilters.

We also note that any homomorphism $\phi: S \to S$ with $\phi(S) \subseteq \Lambda(\beta S)$ has a unique continuous extension $\tilde{\phi}: \beta S \to \beta S$ which is also a homomorphism.

Theorem 4.1 For a discrete commutative semigroup S, if $\phi: S \to S$ is a homomorphism and $p, q \in \beta S$, then if $p \leq_{RK} q$ then $\tilde{\phi}(p) \leq_{RK} q$.

Proof: Since $p \leq_{RK} q$, there is a mapping $f: S \to S$ such that $\tilde{f}(q) = p$. Now $\phi \circ f$ is a continuous mapping from S to S and $\widetilde{\phi \circ f}(s) = (\phi \circ f)(s) = (\phi \circ \tilde{f})(s)$ $\forall s \in S$. Therefore $\widetilde{\phi \circ f} = \tilde{\phi} \circ \tilde{f}$, which implies $\widetilde{\phi \circ f}(q) = (\tilde{\phi} \circ \tilde{f})(q) = \tilde{\phi}(p)$. Consequently $\tilde{\phi}(p) \leq_{RK} q$.

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Corollary 4.2 For a discrete commutative semigroup S, if $\phi: S \to S$ is a homomorphism and $p, q \in \beta S$, then if $p \approx_{RK} q$ then $\tilde{\phi}(p) \leq_{RK} q$ and $\tilde{\phi}(q) \leq_{RK} p$.

For every $s \in S$, using the continuity of the map $\widetilde{\lambda_s}: \beta S \to \beta S$ we can prove the following result:

Corollary 4.3 For a discrete commutative semigroup S, if $p \approx_{RK} q$ for $p, q \in \beta S$, then $s + p \leq_{RK} q$ and $s + q \leq_{RK} p$ for every $s \in S$.

Definition 4.4 For a discrete semigroup S, a homomorphism $\phi: S \to S$ is said to commute with bijections if for every bijective mapping $f: S \to S$, $\tilde{\phi} \circ \tilde{f} = \tilde{f} \circ \tilde{\phi}$.

Theorem 4.5 For a discrete commutative semigroup S, if $\phi: S \to S$ is a homomorphism which commutes with bijections and $p, q \in \beta S$, then if $p \approx_{RK} q$ then $\tilde{\phi}(p) \approx_{RK} \tilde{\phi}(q)$.

Proof: Since $p \approx_{RK} q$, by theorem 3.13 there is a bijective mapping $g: S \to S$ such that $\tilde{g}(q) = p$. Now $g \circ \phi$ is a continuous mapping from S to S and $\widetilde{g \circ \phi}(s) = (g \circ \phi)(s) = (g \circ \tilde{\phi})(s)$ $\forall s \in S$. Therefore $\widetilde{g \circ \phi} = \tilde{g} \circ \tilde{\phi}$, which implies $\widetilde{g \circ \phi}(q) = (\tilde{g} \circ \tilde{\phi})(q) = (\tilde{\phi} \circ \tilde{g})(q) = \tilde{\phi}(p)$. This implies $\tilde{\phi}(p) \leq_{RK} \tilde{\phi}(q)$. Interchanging p and q and using the theorem 3.13 we can prove that $\tilde{\phi}(q) \leq_{RK} \tilde{\phi}(p)$. Consequently $\tilde{\phi}(p) \approx_{RK} \tilde{\phi}(q)$

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