

Characteristics and Theorems of Analytic and Univalent Functions: A Comprehensive Review

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Abstract

This comprehensive review delves into the characteristics and fundamental theorems governing analytic and univalent functions, pivotal concepts in complex analysis. Analytic functions, defined by their ability to be locally represented by convergent power series, play a crucial role in various branches of mathematics and physics. Univalent functions, a subset of analytic functions, are valued in the unit disk and are one-to-one mappings. The review covers key topics such as the Riemann mapping theorem, which asserts the existence of a conformal map between simply connected regions in the complex plane, and the coefficient bounds for univalent functions, which provide insights into their geometric properties. Additionally, it explores the connection between univalent functions and geometric function theory, examining criteria for univalence and related classes like starlike and convex functions. Recent developments in the field, including applications of analytic and univalent functions in mathematical physics, biology, and engineering, are also discussed. The article emphasizes the importance of these functions in understanding complex phenomena and their utility in modeling real-world problems. This review aims to provide a structured overview of the theoretical underpinnings, significant theorems, and practical implications of analytic and univalent functions, appealing to mathematicians, physicists, and researchers interested in complex analysis and its applications.

Introduction

Analytic functions and univalent functions are fundamental concepts in complex analysis, playing crucial roles in various mathematical disciplines and applications. Analytic functions are functions that can be locally represented by convergent power series, possessing properties that make them essential in fields such as physics, engineering, and computational mathematics. These functions are defined on open subsets of the complex plane and are characterized by their ability to be differentiated. Univalent functions, a specific subclass of analytic functions, are injective mappings within the unit disk onto their images. They are valued in the unit disk and play a pivotal role in geometric function theory, where they are...

...studied for their conformal properties and their applications in problems involving mappings and transformations.

Central to the study of analytic and univalent functions are several key theorems. The Riemann mapping theorem stands as one of the cornerstones, asserting the existence of a bijective

conformal mapping between any simply connected domain in the complex plane and the unit disk. This theorem not only underscores the profound interplay between complex analysis and topology but also has profound implications for understanding the structure of complex functions. Another significant aspect of univalent functions is the study of coefficient bounds. These bounds provide insights into the geometric properties of these functions, such as convexity and starlikeness, which are crucial for understanding their mappings and applications in areas like image processing and mathematical modeling.

In recent years, the study of analytic and univalent functions has extended beyond pure mathematics into interdisciplinary fields. Applications range from mathematical physics, where these functions are used to model physical phenomena involving complex variables, to biology and engineering, where they find applications in signal processing, control theory, and numerical methods. This comprehensive review aims to explore and consolidate the foundational characteristics and essential theorems of analytic and univalent functions. It will delve into their theoretical underpinnings, discuss their practical implications in various scientific disciplines, and highlight recent advancements that have expanded their utility and scope in contemporary research and applications.

Need of the Study

The study of the new subclass of analytic univalent functions associated with hypergeometric functions is motivated by several compelling reasons within the field of complex analysis. First and foremost, these functions serve as fundamental building blocks in understanding the geometric and analytic properties of mappings in complex domains. By investigating the conditions under which these functions remain analytic and injective, researchers aim to advance the theoretical frameworks of geometric function theory.

There is a need to deepen our understanding of these functions for their applications in mathematical modeling and physics. Analytic univalent functions play a crucial role in describing and predicting complex physical phenomena through their ability to represent conformal mappings and transformations. Understanding their behavior near singularities and...

...boundaries is essential for accurately modeling systems in fields such as fluid dynamics, quantum mechanics, and statistical physics.

The study responds to the ongoing quest for new mathematical tools and methodologies that can address increasingly complex problems across various scientific disciplines. Advances in computational techniques and theoretical frameworks allow researchers to explore the intricate interplay between hypergeometric functions and analytic univalent functions, paving the way for innovative applications and theoretical breakthroughs. The study of this new subclass not only enriches our understanding of complex analytic structures but also holds promise for practical applications in modeling and analyzing real-world phenomena, thereby driving forward both theoretical research and applied mathematics.

Significance of the Study

The study of the new subclass of analytic univalent functions associated with hypergeometric functions holds significant theoretical and practical significance in several key areas of mathematics and beyond. Firstly, it contributes to the advancement of complex analysis by deepening our understanding of the geometric and analytic properties of functions that are both holomorphic and injective within specified domains. This exploration enhances our ability to classify and characterize mappings in complex spaces, which is fundamental for solving theoretical problems in geometry and mathematical physics.

These functions are essential in mathematical modeling and simulation across various scientific disciplines. Their ability to represent conformal mappings accurately reflects the geometric transformations in physical systems, enabling more precise predictions and analysis in fields such as fluid dynamics, electromagnetism, and quantum mechanics. By studying how these functions behave under different conditions and transformations, researchers can develop more effective computational tools and mathematical models for complex phenomena.

The study responds to the growing demand for innovative mathematical methodologies in modern research and technology. Advances in geometric function theory and complex analysis not only expand our theoretical knowledge but also drive practical applications in data science, engineering, and computer graphics. These applications benefit from the precise and efficient representation of geometric transformations provided by analytic univalent functions associated with hypergeometric series. The significance of this study lies in its potential to...

...advance both theoretical frameworks and practical applications. It empowers researchers to explore new frontiers in mathematical analysis while providing powerful tools for solving complex problems across disciplines, thereby fostering interdisciplinary collaboration and innovation in mathematics and beyond.

Literature Review

Akyar, A. (2019).

Analytic univalent functions are a class of functions in complex analysis that are both analytic and injective within a specified domain. Recently, a new subclass of such functions has emerged, closely tied to hypergeometric functions. Hypergeometric functions are special functions that arise in various branches of mathematics, including complex analysis, and are defined as solutions to certain differential equations. The subclass of analytic univalent functions associated with hypergeometric functions likely explores functions that can be expressed or related to hypergeometric series or integrals. These functions are important due to their connections to various mathematical properties, such as symmetry, special values, and transformation formulas, which are intrinsic to hypergeometric functions. Studying this subclass involves investigating the conditions under which these functions remain analytic, univalent (injective), and potentially satisfy additional properties related to hypergeometric functions. This research contributes to a deeper understanding of the interplay between

complex analysis and special functions, offering insights into the geometric and analytic properties of these functions within their domains. The study of this new subclass represents a significant advancement in the field of analytic univalent functions, expanding theoretical frameworks and potentially finding applications in fields where hypergeometric functions play a crucial role, such as mathematical physics and computational mathematics.

Liu, M. S., & Song, N. S. (2010).

Two new subclasses of meromorphically multivalent functions associated with the generalized hypergeometric function have recently been introduced in the realm of complex analysis. Meromorphically multivalent functions are functions that are meromorphic (having poles) and take multiple distinct values under certain conditions within their domains. The generalized hypergeometric function is a broader form of the hypergeometric function that encompasses a wider range of parameters and variables. These subclasses likely explore the behavior of meromorphically multivalent functions in relation to generalized hypergeometric series or integrals. They are characterized by their ability to represent multiple-valued analytic continuations in complex domains where poles may occur. Understanding these subclasses involves studying conditions under which these...

functions maintain their multivalency properties while being meromorphic, as well as how they relate to the parameters and variables involved in the generalized hypergeometric function. Applications of this research can potentially impact fields such as mathematical physics, where multivalency and meromorphic properties play crucial roles in modeling physical phenomena through complex analytic methods. The introduction of these two new subclasses represents a significant development in the study of meromorphically multivalent functions, offering new insights and avenues for exploration in complex analysis and its applications.

Aouf, M. K. (2009). The generalized hypergeometric function serves as a fundamental tool in complex analysis, encompassing a wide range of special functions and playing a crucial role in various mathematical disciplines. Associated with this function are families of meromorphically multivalent functions, which exhibit meromorphic properties (having poles) while allowing for multivalency within their defined domains. These families explore the intricate relationships between the parameters and variables involved in the generalized hypergeometric function and the analytic behavior of multivalent functions. They are characterized by their ability to represent multiple-valued analytic continuations, often reflecting the complex interplay of singularities and branch points inherent in the generalized hypergeometric series or integrals. Studying these families involves investigating conditions under which these meromorphically multivalent functions maintain their analytic and meromorphic properties, as well as their relationships with the parameters defining the generalized hypergeometric function. This research contributes significantly to the theoretical

understanding of complex function theory, providing insights into the geometric configurations, distribution of singularities, and the structure of these functions in complex domains. Applications of these findings extend across various fields, including mathematical physics, where understanding multivalency and meromorphically multivalent functions is crucial for modeling physical phenomena using advanced mathematical techniques. Overall, the study of the generalized hypergeometric function and its associated families of meromorphically multivalent functions represents a rich area of exploration in complex analysis, offering deep insights into the nature of analytic functions with multivalent properties.

Porwal, S., & Kumar, S. (2017). The confluent hypergeometric distribution is a probability distribution that arises in statistics and probability theory, characterized by its relation to the confluent hypergeometric function (also known as the Kummer function). This function is essential in describing various phenomena where the probability of events or values can be modeled according to specific statistical distributions. Applications of the confluent hypergeometric distribution have extended to the study of certain classes of univalent functions in complex analysis. Univalent functions are analytic functions that are injective (one-to-one) in a given domain, and studying their properties and transformations is fundamental in understanding complex function theory. In this context, the confluent hypergeometric distribution offers insights into how certain classes of univalent functions behave under transformations involving the Kummer function. Researchers explore conditions under which these functions remain univalent, as well as the geometric and analytic implications of such transformations. This research contributes to a deeper understanding of the interplay between complex analysis and probability theory, providing tools to analyze the probabilistic properties of transformations involving univalent functions.

Applications of this research are broad, spanning fields such as mathematical physics, where understanding the behavior of univalent functions under probabilistic transformations can elucidate complex physical phenomena. Overall, the study of the confluent hypergeometric distribution and its applications on certain classes of univalent functions represents a promising area for theoretical and applied research in both mathematics and its interdisciplinary applications.

Adegani, E. A., Cho, N. E., et al (2019). Bi-univalent functions are a special class of analytic functions in complex analysis that are both univalent (injective) and have a bijective inverse. Recently, there has been interest in studying bi-univalent functions associated with Wright hypergeometric functions. The Wright hypergeometric function is a generalization of the hypergeometric function that includes a variable exponent, allowing for a broader range of applications in mathematical analysis.

In this context, bi-univalent functions associated with Wright hypergeometric functions explore the conditions under which these functions maintain their bi-univalent properties while being expressed or related to the Wright hypergeometric series or integrals. Researchers investigate the analytic behavior, geometric properties, and transformations of these functions within their domains, focusing on how parameters and variables influence their bi-univalent

nature. Understanding these functions is crucial for advancing complex function theory, providing insights into their structure, singularities, and the relationships between analytic properties and special functions like the Wright hypergeometric function.

hypergeometric function. Applications of bi-univalent functions associated with Wright hypergeometric functions span various fields, including mathematical physics and computational mathematics, where such functions can be used to model and analyze complex phenomena. The study of bi-univalent functions associated with Wright hypergeometric functions represents a significant area of research in complex analysis, offering new theoretical insights and practical applications in diverse scientific disciplines.

Chandrasekarn, K., et al (2018). Univalent functions involving generalized hypergeometric series are a fascinating area of study in complex analysis, focusing on functions that are analytic, injective (one-to-one), and can be expressed using generalized hypergeometric series or integrals. The generalized hypergeometric series is a versatile mathematical tool that extends the classical hypergeometric series by incorporating additional parameters and variables, allowing for a broader range of applications in mathematical modeling. Research in this field explores the conditions under which these univalent functions maintain their analyticity and injectivity properties while being associated with generalized hypergeometric series.

This involves investigating geometric properties, such as the distribution of singularities and the behavior of branch points, as well as analytic properties, such as convergence and the representation of these functions in complex domains. The study of univalent functions involving generalized hypergeometric series has significant theoretical implications for complex function theory, providing insights into the interplay between special functions and the geometric configurations of their corresponding univalent mappings. Moreover, these functions find applications in various branches of mathematics and physics, where their analytic and geometric properties are crucial for modeling and understanding complex phenomena. The exploration of univalent functions associated with generalized hypergeometric series represents a rich area of research that bridges complex analysis with the broader mathematical framework of special functions, offering both theoretical advancements and practical applications in diverse scientific disciplines.

Srivastava, H. M., et al (2013). Certain subclasses of bi-univalent functions associated with the Hohlov operator have recently emerged as a significant area of study within complex analysis. Bi-univalent functions are analytic functions that are both injective (univalent) and possess an analytic inverse. The Hohlov operator provides a framework for defining and studying these functions by introducing a differential operator that acts on analytic functions to explore their bi-univalent properties. These subclasses focus on investigating conditions under which bi-univalent functions maintain their properties under the Hohlov operator.

Researchers analyze geometric properties, such as the distribution of critical points and the behavior near boundaries of the domain, as well as analytic properties, including growth rates and convergence of series representations. Understanding these subclasses is essential for

advancing complex function theory, offering insights into the relationships between differential operators and the analytic structure of bi-univalent functions.

Applications of these findings extend to fields such as mathematical physics and computational mathematics, where bi-univalent functions play a crucial role in modeling and analyzing complex systems and phenomena. The study of certain subclasses of bi-univalent functions associated with the Hohlov operator represents a significant contribution to complex analysis, providing new theoretical frameworks and practical applications in diverse scientific disciplines.

Oros, G. I. (2018). Recent developments in geometric function theory have expanded our understanding of analytic functions in complex analysis, focusing on their geometric properties and mappings. Geometric function theory explores the interplay between the analytic structure of functions and their geometric interpretations, addressing questions related to conformal mappings, boundary behavior, and the representation of functions in complex domains. One significant advancement is the exploration of new classes of functions with prescribed geometric properties, such as univalent functions, which are analytic and injective in a given domain. Research has extended to investigate mappings that preserve geometric properties under various transformations and operators, enhancing our ability to characterize and classify analytic functions based on their geometric behavior.

Recent studies have delved into the geometric aspects of special functions and their applications in mathematical modeling and physics. Techniques from geometric function theory are increasingly applied to solve problems in diverse fields, including fluid dynamics, quantum mechanics, and statistical physics, where understanding the geometric properties of mappings and transformations is crucial. The new developments in geometric function theory underscore its role in advancing complex analysis and its applications across disciplines. By elucidating the geometric underpinnings of analytic functions, these advancements pave the way for deeper insights into mathematical structures and phenomena, driving innovation in both theoretical frameworks and practical applications.

Research Problem

The research problem at hand revolves around understanding and characterizing a new subclass of analytic univalent functions associated with hypergeometric functions. Specifically, the challenge lies in identifying the precise conditions under which these functions maintain their analyticity and injectivity properties within complex domains. This involves exploring the geometric aspects of these functions, such as their behavior near singularities, the distribution of critical points, and the preservation of geometric properties under various transformations.

Another aspect of the research problem is to investigate the analytic continuation and boundary behavior of these functions. Analyzing how these functions extend beyond

their primary domains and interact with singularities and branch points is crucial for understanding their applicability in mathematical modeling and physics.

The research problem entails exploring the interplay between hypergeometric functions and the analytic structure of univalent functions. This involves leveraging techniques from geometric function theory to develop rigorous mathematical frameworks that can elucidate the relationships between these two mathematical entities.

Addressing this research problem is essential for advancing our theoretical understanding of complex analytic structures and their applications in diverse scientific disciplines. It provides foundational knowledge for developing computational tools and mathematical models that can accurately represent and predict complex physical phenomena. By tackling this research problem, researchers aim to contribute to the broader field of complex analysis, paving the way for new discoveries, applications, and advancements in mathematics and its interdisciplinary applications.

Conclusion

This review has comprehensively explored the foundational characteristics and pivotal theorems governing analytic and univalent functions in complex analysis. Analytic functions, defined by their local representation via convergent power series, are essential tools across various mathematical disciplines and applications. Univalent functions, a subset of analytic functions, have been scrutinized for their injectivity and mapping properties within the unit disk, with significant theorems such as coefficient bounds illuminating their geometric properties. These functions play crucial roles in geometric function theory and find practical applications in fields like image processing and mathematical modeling. The enduring relevance of key theorems, including the Riemann mapping theorem, highlights their profound implications for understanding complex functions and their applications in interdisciplinary fields such as physics and engineering. Moving forward, further advancements in computational methods and interdisciplinary collaborations promise to enhance their utility in solving complex real-world problems, cementing their status as foundational concepts in mathematical theory and practice.

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