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Fractional Delay Differential Equations Using Elzaki Transform

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Abstract

Inthis,we define the Elzaki transform for Delay differential equations using fractional derivative. Here, a Caputo Fractional derivative is also used. As we know the transformation is not easy for finding out to different type of delay differential equations with derivatives of order and their integrals. Therefore we are going to work on Fractional Delay Differential Equations (FDDEs) with transform. The result shows that the method converges to exact solution.

Keywords

Fractional delay differential equations, Elzaki transform, Caputo derivative, ElzakiAdomian Decomposition method, Numerical solution.

Introduction

Usingmathematical models there are so many real problems that can be explained. As we knowmathematical model is a simplified description of physical reality expressed in mathematical terms. Thus, the investigation of the exact or approximation solution helps us to understand the means of these mathematical models. Several numerical methods were developed for solving ordinary or partial differential equations. New integral transform Elzaki transform is particularly useful for finding solutions for Fractional delay differential equation (FDDE's). Elzaki transform is a useful technique for solving linear delay differential equations these equations but this transform is totally incapable of handling nonlinear equations because of the difficulties that are caused by the nonlinear terms. [1, 3–7]. But with the help of Fractional delay differential equations (FDDEs) we can use in modeling certain processes and systems in

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

engineering and other science. Aim of this work is to find out the approximate solution of fractional delay differential equations using Elzaki transform and adomian decomposition methods [9]. This paper is divided in to three parts. In part1 there is introduction part. In part 2 there are some definitions and properties related to Elzaki Transform and method of solution is presented. Part 3 defines concepts related to fractional delay differential equations. In last part there are some examples using Elzaki transform method with delay differential equations of fractional order, with illustrative examples have been given.

2. The Elzaki Transform

There are some integral transform that are defined in thedomain $t \geq 0$, where t represents time such as Laplace transform ,Sumudu transform, the Natural transform, Aboodh transform respectievely [4]. So, here we define a newtransform which is Elzaki transform is defined for functions of exponential order. Let us consider the set A ;

$$A = \left\{ f(t) : |f(t)| < M e^{|t|/k_j}, \text{ if } t \in (-1)^j \times [0, \infty) \ , j = 1, 2 \ ; \varepsilon_i > 0 \right\}$$

where $\varepsilon_1, \varepsilon_2$ may be finite or infinite.

Then, the operator E (.) denotes Elzaki transform denoted by is defined by integral equation:

$$E[f(t)] = T(u) = u \int_0^\infty f(t) e^{\frac{-t}{u}} dt \ , t \geq 0 \ , \ \varepsilon_1 \leq u \leq \varepsilon_2 \qquad (2.1)$$

Here,the variable u is used to express the variable t in the form of the function f .

Theorem 2.1 Partial derivatives of some Elzaki transform:

1. $E[f'(t)] = \frac{T(u)}{u} - uf(0)$
2. $E[f''(t)] = \frac{T(u)}{u^2} - f(0) - uf'(0)$

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
Research paper Volume 11, Iss 10, 2022

3. $$E\left[f^{n}(t)\right]=\frac{T(u)}{u^{n}}-\sum_{k=0}^{n-1}u^{2-n+k}f^{k}(0) \tag{2.2}$$

$f(t)$	$T(u)=E\left[f(t)\right]$
1	u^2
t	u^3
t^n ; $n\geq 1$	$n!u^{n+2}$
e^{at}	$\frac{u^2}{1-au}$
$\sin at$	$\frac{au^3}{1+a^2u^2}$
$\cos at$	$\frac{u^2}{1+a^2u^2}$
$\sinh at$	$\frac{au^3}{1-a^2u^2}$
$\cosh at$	$\frac{u^2}{1-a^2u^2}$

1.

2.1. Elzaki transform method:-

Consider the general nonlinear ordinary differential equation (ODE) of the form[8]:

$$\frac{d^{n}y(t)}{dt^{n}}+P(y)+Q(t-\tau)=g(t) \;\; ; n=1,2,3,\ldots \tag{2.3}$$

With initial condition

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

$$y^{k}(0) = y_{0}^{k} \tag{2.4}$$

where, $\frac{d^{n}y(t)}{dt^{n}}$ represents derivative of y oforder n. P and Q are linear and nonlinear bounded operator respectivelyand $g(t)$ denotes the continuous function, and y is a function of t which is of the form as , $y = y(t)$.

As it is defined earlier by Wu [4] , for identifying the Lagrange multiplier in easily way we have to apply the Elzaki transform on both sides of equation(2.3)-(2.4) in such a manner that an equation containing linear part with constant coefficients is transformed in to algebraic one . Now, using this method we can easily find out the unknown Lagrange multiplier. Now using, Elzakitransform on both sides of (2.3) and (2.2) such that the linear part is transferred into an algebraic equation as:

On taking the Elzaki transform, we get,

$$E\left[\frac{d^{n}y(t)}{dt^{n}}\right] + E\left[P(y)\right] + E\left[Q(t-\tau)\right] = E\left[g(t)\right] \tag{2.5}$$

But,

$$E\left[\frac{d^{n}y(t)}{dt^{n}}\right] = \frac{E(y(t))}{u^{n}} - Au^{2-n+k} \tag{2.6}$$

where, $A = \sum_{k=0}^{n-1} g^{k}(0)$

$$E(y(t)) = Au^{2+k} - u^{n}\left[P(y)\right] - u^{n}\left[Q(t-\tau)\right] + u^{n}\left[g(t)\right] \tag{2.7}$$

Now, Elzaki decomposition method for solution y(t) is defined by series :

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

$$y(t) = \sum_{n=0}^{\infty} y_n(t) \tag{2.8}$$

the nonlinear operator is defined as:

$$Q(t-\tau) = \sum_{n=0}^{\infty} B_n \tag{2.9}$$

where B_n represents adomain polynomial of $y_0, y_1, y_2, \ldots\ldots, y_n$ which is given by:

$$B_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\left[Q\left(\sum_{n=0}^{\infty} \lambda^n y_n\right)\right]$$

Then the Adomian series follows as:

$$B_0 = f(y_0)$$

$$B_1 = y_1 f'(y_0)$$

$$B_2 = y_2 f'(y_0) + \frac{y_1^2}{2!} f''(y_0)$$

$$B_3 = y_3 f'(y_0) + y_1 y_2 f''(y_0) + \frac{y_1^3}{3!} f'''(y_0) \tag{2.10}$$

:

:

Using (2.8) and (2.9) into (2.7), we get

$$E\left(\sum_{n=0}^{\infty} y_n\right) = Au^{2+k} - u^n E\left[P\left(\sum_{n=0}^{\infty} y_n\right)\right] - u^n E\left[\sum_{n=0}^{\infty} B_n\right] + u^n\left[g(t)\right] \tag{2.11}$$

Equating both side of equation (2.11)

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
Research paper Volume 11, Iss 10, 2022

$$E[y_0] = Au^{2+k} + u^n E[g(t)] \tag{2.12}$$

$$E[y_1] = -u^n E[Py_0] - u^n E[B_0] \tag{2.13}$$

$$E[y_2] = -u^n E[Py_1] - u^n E[B_1] \tag{2.14}$$

Now the recursive relation is given by:

$$E\left[y_n\right] = -u^n E\left[Py_{n-1}\right] - u^n E\left[B_{n-1}\right] \quad ; n \geq 1 \tag{2.15}$$

Using Elzaki inverse transform on (2.12)-(2.15), we get:

$$y_0 = K(t)$$

$$y_n = -E^{-1}\left[u^n E\left[Py_{n-1}\right]\right] - E^{-1}\left[u^n E\left[B_{n-1}\right]\right] \quad ; n \geq 1$$

Where, K(t) represents the function which satisfies the initial conditions.

2.2. Some Illustrative problems:

Problem2.2.1 Let us consider nonlinear delay differential equation (NDDE) :

$$y'(t) = 1 + 2y^2\left(\frac{t}{2}\right) \quad , 0 \leq t \leq 1 \tag{2.16}$$

Using initial condition, $y(0) = 0$.

Exact solution of the given problem is:

$$y(t) = \sinh t$$

Taking Elzaki transform on both sides, we get :

$$E\left[y'(t)\right] = E[1] + 2E\left[y^2\left(\frac{t}{2}\right)\right]$$

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

Now using definition (ii) in theorem (2.1), we get:

$$E\left[y'(t)\right]=\frac{T(u)}{u}-uf(0)=u^2+2E\left[y^2\left(\frac{t}{2}\right)\right]$$

So, we have

$$T(u)=E\big(y(t)\big)=u^3+2uE\left[y^2\left(\frac{t}{2}\right)\right] \tag{2.17}$$

Now taking inverse of Elzaki operator, E^{-1}, on both sides of (2.17), we have:

$$y(t)=E^{-1}\left[u^3\right]+2E^{-1}\left[uE\left(y^2\left(\frac{t}{2}\right)\right)\right] \tag{2.18}$$

Using the Table 1, we have

$E^{-1}\left[u^3\right]=t$;andhence we have

$$y_0\left(\frac{t}{2}\right)=\frac{t}{2}$$

$$y_{n+1}=2E^{-1}\left[uE\left[B_n\right]\right] \tag{2.19}$$

Using equation (2.10), we get;

$$B_0=f(y_0)=y_0^2\left(\frac{t}{2}\right)$$

$$B_1=y_1f'(y_0)=2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)$$

$$B_2=y_2\left(\frac{t}{2}\right)2y_0\left(\frac{t}{2}\right)+y_1{}^2\left(\frac{t}{2}\right)$$

:

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

:

Now the equation (2.19), become at n=0 :

$$y_1(t) = 2E^{-1}\left[uE\left[B_0\right]\right] = 2E^{-1}\left[uE\left[\frac{t^2}{4}\right]\right] = 2E^{-1}\left[\frac{u^5}{2}\right] = \frac{t^3}{3!}$$

$$y_1\left(\frac{t}{2}\right) = \frac{t^3}{48}$$

Equation (2.19) becomes for n=1:

$$y_2(t) = 2E^{-1}\left[uE\left[B_1\right]\right] = 2E^{-1}\left[uE\left[2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right]\right] = 2E^{-1}\left[uE\left[\frac{t^4}{48}\right]\right] = \frac{t^5}{5!}$$

$$y_2\left(\frac{t}{2}\right) = \frac{t^5}{3840}$$

Equation (2.19) , becomes for n=2:

$$y_3(t) = 2E^{-1}\left[uE\left[B_2\right]\right] = 2E^{-1}\left[uE\left[2y_2\left(\frac{t}{2}\right)y_0\left(\frac{t}{2}\right)+y_1^2\left(\frac{t}{2}\right)\right]\right] = 2E^{-1}\left[uE\left[\frac{t^6}{3840}+\frac{t^6}{2304}\right]\right] = \frac{t^7}{7!}$$

Therefore the approximate solution is given as:

$$y(t) = y_0(t)+y_1(t)+y_2(t)+\dots = \mathrm{t}+\frac{t^3}{3!}+\frac{t^5}{5!}+\frac{t^7}{7!}+\dots = \sinh t.$$

3. Delay Differential Equation in Fractional Form:

Here we use Elzaki decomposition method for solving fractional delay differential equation in linear and non linear form.

Definition3.1. Caputo fractional derivative of Elzaki Transform is defined as [2]:

$$E\left[D^{\alpha} f(t)\right] = \frac{1}{u^{\alpha}} E\left[f(t)\right] - \sum_{k=0}^{n-1} u^{-\alpha+2+k} f^{k}(0)$$

$$E\left[t^{\alpha}\right]=\Gamma(\alpha+1)u^{\alpha+2}$$

3.2 Analyization of Method:

Let us take general nonlinear ordinary differential equation of the form:

$$D^{\alpha}y(t)+P(y)+Q(t-\tau)=g(\mathrm{t}) \;\; ; \; \tau\in R \; ; \; t<\tau, n-1\leq\alpha\leq n \tag{3.1}$$

With initial condition:

$$\mathrm{y}^{k}(0)=y_{0}^{k}$$

where $D^{\alpha}y(t)$ is the term of the fractional order derivative , P and Q are linear and nonlinear bounded operator respectively and continuous function is defined in terms of $g(\mathrm{t})$ and y is as function of t which is denoted as $y=y(t)$. The Elzakidecomposition method consists of applying the Elzaki transform first on both sides of(3.1), to give:

$$E\left[D^{\alpha}f(t)\right]+E\left[P(y)\right]+E\left[Q(t-\tau)\right]=E\left[g(\mathrm{t})\right]$$

By definition (3.1)

$$E(y(t))=Au^{2+k}-u^{\alpha}\left[P(y)\right]-u^{\alpha}\left[Q(t-\tau)\right]+u^{\alpha}\left[g(t)\right] \tag{3.2}$$

where, $A=\sum_{k=0}^{n-1}g^{k}(0)$

Elzaki decomposition method defines the solution for $y(t)$ using series:

$$y(t)=\sum_{n=0}^{\infty}y_{n}(t) \tag{3.3}$$

Nonlinear operator is defined as:

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

$$Q(t-\tau) = \sum_{n=0}^{\infty} B_n \tag{3.4}$$

where, B_n is defined in (2.9). As we know earlier, first Adomnian polynomials are given in (2.10).

Using (3.3) and (3.4) in (3.2), we get:

$$E\left[\sum_{n=0}^{\infty} y_n(t)\right] = Au^{2+k} - u^{\alpha} E\left[P\left(\sum_{n=0}^{\infty} y_n(t)\right)\right] - u^{\alpha} E\left[\sum_{n=0}^{\infty} B_n\right] + u^{\alpha} E\left[g(t)\right] \tag{3.5}$$

Equating both sides of (3.5):

$$E[y_0] = Au^{2+k} + u^{\alpha} E\left[g(t)\right] \tag{3.6}$$

$$E[y_1] = -u^{\alpha} E[Py_0] - u^{\alpha} E[B_0] \tag{3.7}$$

$$E[y_2] = -u^{\alpha} E[Py_1] - u^{\alpha} E[B_1] \tag{3.8}$$

Therefore the recursive relation is defined by

$$E\left[y_n\right] = -u^{\alpha} E\left[Py_{n-1}\right] - u^{\alpha} E\left[B_{n-1}\right] \quad ; n \geq 1 \tag{3.9}$$

Taking inverse Elzaki transform to (3.6)-(3.8), we get:

$$y_0 = K(t)$$

$$y_n = -E^{-1}\left[u^{\alpha} E\left[Py_{n-1}\right]\right] - E^{-1}\left[u^{\alpha} E\left[B_{n-1}\right]\right] \quad ; n \geq 1 \tag{3.10}$$

Here K (t) denotes a function which satisfies the initial conditions.

3.2. Illustrative problems:

Problem3.2.1. Let us consider the Fractional nonlinear delay differential equation (NDDE) is of the form

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
Research paper Volume 11, Iss 10, 2022

$$D^{\alpha} y(t) = 1 + 2y^2\left(\frac{t}{2}\right) \quad , 0 \le t \le 1 ; 0 < \alpha \le 1 \tag{3.11}$$

with initial condition

$$y(0) = 0$$

Taking Elzaki transform to both sides of equation (3.9), we get

$$E\left[D^{\alpha} y(t)\right] = E\left[1 + 2y^2\left(\frac{t}{2}\right)\right]$$

By definition (3.1) and using initial condition (3.10), we have:

$$u^{-\alpha} E\left[y(t)\right] = u^2 + E\left[2y^2\left(\frac{t}{2}\right)\right]$$

$$E\left[y(t)\right] = u^{2+\alpha} + u^{\alpha} E\left[2y^2\left(\frac{t}{2}\right)\right] \tag{3.12}$$

$$y(t) = E^{-1}\left[u^{2+\alpha}\right] + E^{-1}\left[u^{\alpha} E\left[2y^2\left(\frac{t}{2}\right)\right]\right]$$

Where, $y_0(t) = E^{-1}\left[u^{2+\alpha}\right] = \dfrac{t^{\alpha}}{\Gamma(\alpha+1)}$

So, we get:

$$y_0\left(\frac{t}{2}\right) = \frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}$$

And also we have;

$$y_{n+1}(t) = E^{-1}\left[u^{\alpha} E\left[2B_n\right]\right] \tag{3.13}$$

Using equation (2.10), we have:

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
Research paper Volume 11, Iss 10, 2022

$$B_0 = y_0^2\left(\frac{t}{2}\right)$$

$$B_1 = 2y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)$$

$$B_2 = y_2\left(\frac{t}{2}\right)2y_0\left(\frac{t}{2}\right)+y_1{}^2\left(\frac{t}{2}\right)$$

Equation (3.13), becomes when n=0 :

$$y_1(t) = E^{-1}\left[u^{\alpha}E\left[2B_0\right]\right] = E^{-1}\left[u^{\alpha}E\left[2y_0^2\left(\frac{t}{2}\right)\right]\right]$$

$$= E^{-1}\left[u^{\alpha}E\left[2\left(\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\right)^2\right]\right] = E^{-1}\left[u^{\alpha}E\left[\left(\frac{t^{2\alpha}}{2^{2\alpha-1}\Gamma^2(\alpha+1)}\right)\right]\right]$$

$$= E^{-1}\left[u^{3\alpha+2}\frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)}\right] = \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)}E^{-1}\left[u^{3\alpha+2}\right]$$

$$= \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

We choose, $C = \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)}$, so, we have :

$$y_1(t) = C\frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$y_1\left(\frac{t}{2}\right) = C\frac{t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)}$$

Equation (3.13) , becomes when n=1:

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

$$y_2(t) = E^{-1}\left[u^{\alpha} E\left[2B_1\right]\right] = E^{-1}\left[u^{\alpha} E\left[4y_0\left(\frac{t}{2}\right)y_1\left(\frac{t}{2}\right)\right]\right]$$

$$= E^{-1}\left[u^{\alpha} E\left[4\left(\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\right)\left(C\frac{t^{3\alpha}}{2^{3\alpha}\Gamma(3\alpha+1)}\right)\right]\right]$$

$$= E^{-1}\left[u^{\alpha} E\left[C\left(\frac{t^{4\alpha}}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right)\right]\right]$$

$$= E^{-1}\left[u^{5\alpha+2} C\frac{\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\right]$$

$$= C\frac{\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)} E^{-1}\left[u^{5\alpha+2}\right]$$

$$= C\frac{\Gamma(4\alpha+1)}{2^{4\alpha-2}\Gamma(\alpha+1)\Gamma(3\alpha+1)}\frac{t^{5\alpha}}{\Gamma(5\alpha+1)}$$

Now series solution is followed by:

$$y(t) = y_0(t)+y_1(t)+y_2(t)+\ldots$$

For $\alpha=1$, we have

$$y_0(t) = \frac{t}{\Gamma(2)} = t$$

$$y_1(t) = \frac{\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)}\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} = \frac{\Gamma(3)}{2^1\Gamma^2(2)}\frac{t^3}{\Gamma(4)} = \frac{t^3}{3!}$$

$$= C\frac{\Gamma(5)}{2^2\Gamma(2)\Gamma(4)}\frac{t^5}{\Gamma(6)} = \frac{t^5}{5!}$$

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

Recall that, $\Gamma(n+1) = n! \quad ,\forall n \in N$

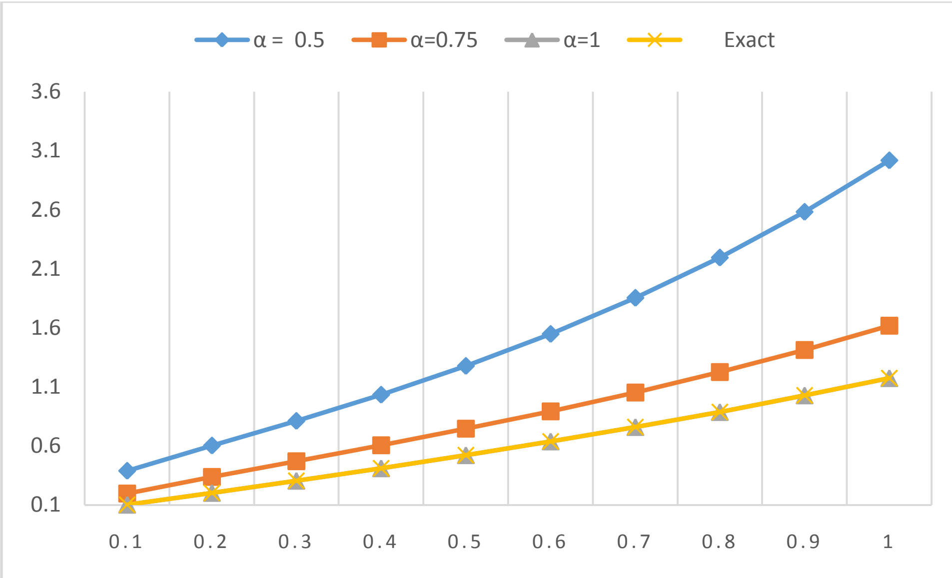
For $\alpha = 1$, the exact solution is :

$$y(t) = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \frac{t^7}{7!} + ... = \sinh t .$$

Table 1: Table 1 represents the approximate solution of Eqn. (17) using ETM for different values of α and the exact solution when $\alpha = 1$.

T	ETM at $\alpha = 0.5$	ETM at $\alpha = 0.75$	ETM at $\alpha = 1$	Exact	Error
0.1	0.3892590717	0.1959585438	0.1001667500	0.1001667500	0.0000000000
0.2	0.6029390047	0.3372919795	0.2013360025	0.2013360025	0.00000000000
0.3	0.8116560351	0.4710909523	0.3045202934	0.3045202934	0.00000000000
0.4	1.0331944433	0.6056522953	0.4107523258	0.4107523258	0.00000000000
0.5	1.2764367496	0.7453582126	0.5210953054	0.5210953054	0.00000000000
0.6	1.5477669356	0.8933439871	0.6366535821	0.6366535821	0.00000000000
0.7	1.8526415018	1.0522633501	0.7585837018	0.7585837018	0.00000000000
0.8	2.1961426891	1.2245995617	0.8881059821	0.8881059821	0.00000000000
0.9	2.5832151481	1.4128212642	1.0265167257	1.0265167257	0.00000000000
1	3.0187803642	1.6194735227	1.1752011935	1.1752011936	0.00000000001

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022



Problem3.2.2. Let us consider nonlinear fractional delay differential equation of n=2

$$D^{\alpha} y(t) = 1 - 2y^2\left(\frac{t}{2}\right) \quad , 0 \le t \le 1 ; 1 < \alpha \le 2 \qquad (3.14)$$

using initial condition

$$y(0) = 1 \quad , y'(0) = 0$$

Taking Elzaki transform on both sides of (3.14), we have

$$E\left[D^{\alpha} y(t)\right] = E\left[1 - 2y^2\left(\frac{t}{2}\right)\right]$$

Now by using equation (3.1) and initial condition

$$u^{-\alpha} E\left[y(t)\right] - u^{-\alpha+2} y(0) - u^{-\alpha+3} y'(0) = u^2 - E\left[2y^2\left(\frac{t}{2}\right)\right]$$

$$E\left[y(t)\right] = u^2 + u^{2+\alpha} - 2u^{\alpha} E\left[y^2\left(\frac{t}{2}\right)\right] \qquad (3.15)$$

Now using inverse Elzaki Transform to (3.15), we have

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

$$y(t)=E^{-1}\left[u^{2}\right]+E^{-1}\left[u^{2+\alpha}\right]-2E^{-1}\left[u^{\alpha}E\left[y^{2}\left(\frac{t}{2}\right)\right]\right]$$

$$y_{0}(t)=E^{-1}\left[u^{2}\right]+E^{-1}\left[u^{2+\alpha}\right]=1+\frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

Now, $y_{0}\left(\frac{t}{2}\right)=1+\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}$

Now, $y_{n+1}(t)=-2E^{-1}\left[u^{\alpha}E\left[B_{n}\right]\right]$ (3.16)

From equation (2.10), we have:

$$B_{0}=y_{0}^{2}\left(\frac{t}{2}\right)=\left(1+\frac{t^{\alpha}}{2^{\alpha}\Gamma(\alpha+1)}\right)^{2}=1+\frac{t^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{2^{2\alpha}\Gamma^{2}(\alpha+1)}$$

$$B_{1}=2y_{0}\left(\frac{t}{2}\right)y_{1}\left(\frac{t}{2}\right)$$

$$B_{2}=y_{2}\left(\frac{t}{2}\right)2y_{0}\left(\frac{t}{2}\right)+y_{1}{}^{2}\left(\frac{t}{2}\right)$$

Now equation (3.16), become when n=0 :

$$y_{1}(t)=-2E^{-1}\left[u^{\alpha}E\left[B_{0}\right]\right]=-2E^{-1}\left[u^{\alpha}E\left[1+\frac{t^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)}+\frac{t^{2\alpha}}{2^{2\alpha}\Gamma^{2}(\alpha+1)}\right]\right]$$

$$=-2E^{-1}\left[u^{\alpha+2}+\frac{u^{2\alpha+2}}{2^{2\alpha-1}}+\frac{\Gamma(2\alpha+1)u^{3\alpha+2}}{2^{2\alpha}\Gamma^{2}(\alpha+1)}\right]$$

$$=-\frac{2t^{\alpha}}{\Gamma(\alpha+1)}-\frac{t^{2\alpha}}{2^{\alpha-2}\Gamma(2\alpha+1)}-\frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^{2}(\alpha+1)\Gamma(3\alpha+1)}+\ldots..$$

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

The series solution is followed by:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots .$$

$$= 1 + \frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{2^{\alpha-2}\Gamma(2\alpha+1)} - \frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \dots..$$

On putting $\alpha = 2$, we get :

$$y_0(t) = 1 + \frac{t^2}{\Gamma(3)} = 1 + \frac{t^2}{2!} = 1 + \frac{t^2}{2}$$

$$y_1(t) = -\frac{2t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{2^{\alpha-2}\Gamma(2\alpha+1)} - \frac{t^{3\alpha}\Gamma(2\alpha+1)}{2^{2\alpha-1}\Gamma^2(\alpha+1)\Gamma(3\alpha+1)} + \dots..$$

$$= -\frac{2t^2}{2!} - \frac{t^4}{2^0\Gamma(5)} - \frac{t^6}{2^3\Gamma^2(3)\Gamma(7)} = -t^2 - \frac{t^2}{4!} - 3\frac{t^6}{4\times 6!}$$

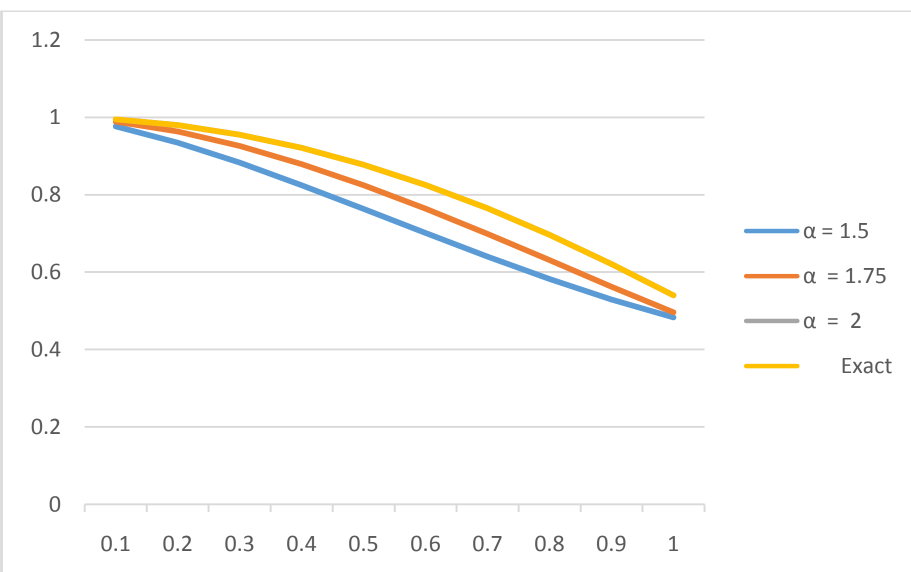
At $\alpha = 2$, the exact solution is:

$$y(t) = y_0(t) + y_1(t) + y_2(t) + \dots = 1 - \frac{t^2}{2!} + \frac{t^4}{4!} + \dots = \cos t$$

Table 2: Table 2 represents the approximate solution of Eqn. (17) using ETM for different values of α and the exact solution when $\alpha = 1$

ISSN PRINT 2319 1775 Online 2320 7876
 Volume 11, Iss 10, 2022

T	$\alpha = 1.5$	$\alpha = 1.75$	$\alpha = 2$	Exact	Error
0.1	0.9764472906	0.9889759328	0.9950041652	0.9950041652	0.0000000000
0.2	0.9346002865	0.9631796010	0.9800665778	0.9800665778	0.0000000000
0.3	0.8827459859	0.9259293031	0.9553364891	0.9553364891	0.0000000000
0.4	0.8247447068	0.8791724851	0.9210609940	0.9210609940	0.0000000000
0.5	0.7634206034	0.8245895622	0.8775825618	0.8775825618	0.0000000000
0.6	0.7011411688	0.7638309012	0.8253356149	0.8253356149	0.0000000000
0.7	0.6400312647	0.6986217817	0.7648421872	0.7648421872	0.0000000000
0.8	0.5820785052	0.6308269209	0.6967067093	0.6967067093	0.0000000000
0.9	0.5291962930	0.5624995596	0.6216099682	0.6216099682	0.0000000000
1	0.48326789280	0.4959242372	0.5403023058	0.5403023058	0.0000000000



ISSN PRINT 2319 1775 Online 2320 7876
Research paper **Volume 11, Iss 10, 2022**

Conclusion: In this study, we proposed a transform for finding out the solutions of fractional delay differential equations. As Adomnian decomposition method has been defined to solve many types of nonlinear differential equations. So this study defines a combination of transform with decomposition method which results as Elzaki decomposition method to solve out for fractional delay differential equations

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