IT IS GENERALLY IMPOSSIBLE TO DETERMINE WHETHER A RATIONAL POINT LIES IN THE INTERIOR OR ON THE BOUNDARY OF A CLOSED SET ALGORITHMICALLY

[1]RAJU BARADI,[2]SRINIVAS BOGA,[3]NARENDRA BANDARI,
[4]THOGARLA PARIMALA
[124]Assistant Professor, [3]Associate Professor
Department of Mathematics
Kshatriya College of Engineering

Abstract:

It seems true that we can almost always determine the position of a specific point when a set is given. Namely, we could assert whether this point lies at this set's interior, boundary, or exterior. However, this is not always the case in constructive mathematics. In this research, we will show that it is generally impossible to algorithmically determine whether a rational point lies in the interior of a closed productive set or on the boundary of it. We conduct our proof by making contradictions. Firstly, We used an unextendible algorithm to construct a rational point and a closed set on the natural line. Secondly, we reformulate the assumption "we could decide whether the point lies in the interior or on the boundary of a closed set" to "we could determine the program will eventually print 1". Thirdly, we constructed an extension of the program to all the positive integers, which is a contradiction to our assumption. Hence, we concluded that it is impossible to figure out the position of the rational point algorithmically.

Keywords: Unextendible Program, Constructive Mathematics, Closed Set, Rational Point.

1. Introduction

Bishop, also known as Albert Bishop, was an American mathematician who contributed significantly to constructive mathematics. Born in 1928 in California, Bishop received his Ph.D. from the University of California, Berkeley, in 1955 and taught at several universities throughout his career. His work in constructive mathematics [1] challenged the traditional approach to mathematics, which relies heavily on the principle of excluded middle and the law of non-contradiction. Instead, Bishop advocated for a more constructive approach to mathematics, emphasizing the importance of constructive proofs and the existence of mathematical objects that can be explicitly constructed. In addition to Bishop, there are also two people that we need to be familiar with. They are Andrei Andreevich Markov Jr. Logic of Markov's school that can be described using basic principles: recursive realizability, Markov principle, and classical logic for sentences containing no constructive problems, see Mints. [2]. Markov often focuses on infinite sequences of natural numbers, using Greek letters for arrangements and Latin letters for individual numbers. Shanin has said that people have been looking for "natural" (from the baseline and idealized point of view of structural mathematics) ways to understand mathematical judgments of different types of constructed objects, see Shanin [3].

"Unlike traditional mathematics, which is concerned with the existence of mathematical objects, constructive

ISSN PRINT 2319 1775 Online 2320 7876
Research Pape Journal Volume 10, Iss 11, 2021

mathematics is concerned with the explicit construction of mathematical objects." Kushner wrote the text.. [4] Constructive mathematics has many practical applications in fields like computer science and logic where explicit construction of mathematical objects is essential. Constructive math focuses on providing evidence of mathematical objects' existence rather than on their existence itself, leading to new insights in areas like topology, analysis, and algebraic geometry (Mandelkern [5]). Constructive mathematics adopts a stricter definition of an algorithm than classical mathematics, defining it as a productive procedure that can generate or construct mathematical objects. This stands in stark contrast to classical mathematics, where an algorithm may simply be an equation used to calculate results without necessarily providing any methodological approach for doing so; an example would be Turing's work [6].

2. Literature Review

Constructive analysis is a subfield of mathematics that emphasizes productive methods and proofs, applying them to mathematical objects and reasoning. Seminal works have contributed to this branch of mathematics foundation and understanding. This literature review provides an overview of key ideas and contributions within each piece reviewed here.

Bishop and Beeson's Foundations of Constructive Analysis is an expansive text that comprehensively introduces constructive analysis. It explores the constructive approach to analysis by emphasizing constructive logic and intuitionistic reasoning, with basic concepts such as constructive logic, set theory, and real numbers covered, as well as continuity, differentiability, and integration from a constructive angle - making this book an indispensable source of knowledge regarding constructive analysis. Kushner's Lectures on Constructive Mathematical Analysis" thoroughly introduces constructive analysis and its applications, covering set theory, logic, real numbers, and topology as examples of constructive mathematics. He emphasizes intuitive sense as an approach to mathematical proofs while offering clear explanations and examples to make his book understandable for beginners and experienced mathematicians interested in constructive mathematics. Mandelkern's article presents an accessible yet in-depth exploration of its central ideas and principles, discussing its motivation and distinguishing it from classical mathematics. Additionally, Mandelkern provides details regarding constructivism, such as the interpretation of logical connectives or existence concepts, as examples of basic constructivist principles he describes within this work. Furthermore, this piece also highlights its application across diverse areas of mathematics, along with the philosophical implications of this form of mathematical thinking. "Stepwise semantics of A. A. Markov." Nauka. Mints' work centers around analyzing the stepwise semantics of A. A. Markov, an esteemed Russian mathematician. Although written entirely in Russian, Mints' contribution enhances our understanding of constructive mathematics by offering insight into the step-by-step construction of mathematical objects and proofs. Her findings add significantly to mathematical logic knowledge and approach towards constructive mathematics approaches. "A Hierarchy of Ways of Understanding Judgments in Constructive Mathematics." Trudy Mat. Inst. Steklov.

eklov. Shanin's article explores various approaches to understanding judgments in constructive mathematics from a hierarchical viewpoint, exploring different proof methods and the meaning of constructive statements. She contributes significantly to our philosophical understanding of constructivism while offering insight into various interpretive frameworks in constructive mathematics. Shen, A. and Vereshchagin, N. K. (2003). Computable Functions. AMS Press.

Shen and Vereshchagin's book "Computable Functions" explores the theory of computable functions as it applies to constructive mathematics. Though not solely dedicated to constructive analysis, this text covers fundamental topics related to computability theory from this constructive angle, providing an indepth exploration of computable functions and various mathematical objects' computability - valuable resources for understanding its constructive aspects. Turing's groundbreaking paper "On computable numbers with an application to the decision problem," published in 1936's Proceedings of London Mathematical Society, laid a firm basis for modern computer science theory and enormously influenced its evolution. This literature review provides an overview of Turing's seminal paper today and its key ideas and contributions presented therein.

Turing introduced his universal computing machine (now commonly referred to as the Turing machine) as a theoretical model of computation in his paper, seeking an answer to David Hilbert's decision problem involving an algorithmic way of deciding the truth or falsity of mathematical statements. Turing's investigation of Hilbert's problem resulted in him coining the term "computability," providing a fundamental understanding of whether difficulties could be solved algorithmically.

3. Definitions

Def 2.1 Constructive Real Number(CRN): Constructive Real Number, also known as CRN, is a combination of two computer programs $\alpha(k)$ and $\beta(k)$, in which $\alpha(k)$ is a sequence of rational numbers and $\beta(k)$ is a sequence of positive integers, such that for $\forall n \in \mathbb{N}$, $|\alpha(p) - \alpha(q)| < 2 - n$ holds for $\forall p, q > \beta(n)$.

Def 2.2 Regulator: Definition 2.1 refers to the computer program as the convergence regulator or convergence risk neutralizer of CRN. A Regulator is a Standard Regulator with the property $\beta(n) = n$ for $\forall n \in \mathbb{N}$.

Def 2.3 Unextendible Program: Unextendible Program is a partially defined computer program that does not terminate for some positive inputs and cannot be extended to another program that works for all positive integer inputs. A classical fact in theoretical Computer Science is that unextendible programs exist; see Shen, A. and Vereshchagin N.K. [7].

Def 2.4 Constructive functions: An algorithm transforms every CRN into a CRN, which should take equivalent CRNs to equivalent CRNs. Markov and Ceitin's (Tzeitin) Theorem says that all constructive functions are continuous; see Kushner B. A. [4].

Remark: Constructive Real Numbers first appeared in a slightly different form in the work of the founder of Computer Science, see Turing A. [6].

4. Notations

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
 Journal Volume 10, Iss 11, 2021

Symbols	Descriptions
E	A closed constructive set
x_0	A rational point
I_n	The n-th closed interval with the rational endpoint
∂E	The boundary of the set
$Int\ E$	The interior of the set
(p, q)	The greatest common divisor of p and q

4. Theorem

It is generally impossible to algorithmically decide whether a rational point on the natural line is in the interior or on the boundary of a closed set. Note that the real line ℝ is precisely the case in 1-dimension, so the heading could be proved if we cannot even assert the position of the rational point in this situation.

5. Proof of the Theorem

Since $x0$ is a rational point, we could denote it as a form of , where p and q are both integers and $q \neq 0$, $(p, q) = 1$. Take an unextendible program P(k), transforming some positive integers to 0 and 1. We define a sequence of closed intervals $In(k)$ as $In(k) = [\ p\ q - 1\ 2\ n\ , M]$ if the program is still working on input k by the n-th step of it being executed or if it stopped working and produced 0, If the program prints 1 at N-th stage, then we define $In(k)$ to be $[\ p\ q - 1\ 2N\ , M]$ for all $n \geq N$(M is a fixed large number). To better illustrate these intervals' construction, we offered the following graphs.

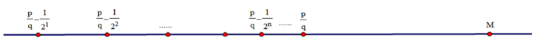

Figure 1. Construction of Intervals where 1 is not yielded.

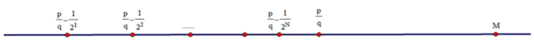

Figure 2. Construction of Intervals where 1 is yielded at the N-th step.

We will prove this theorem by contradiction. Thus, we hope to find an extension of this program if we can decide whether a rational point lies in the interior or the boundary of the closed set , which is the intersection of all $In(k)$ for every fixed k.

The program has several situations. If the program P(k) eventually prints 1, then $p\ q$ is in the interior of our closed set Ik. In all other cases, $p\ q$ is in the boundary of Ik. Assume there is a program Q(I,$p\ q$) that can always decide whether $p\ q$ is always in the interior or on the edge of the closed set I . Apply this program to Ik,   , if Q says $p\ q$ is in the boundary, then we define P'(k) as 0, if Q says $p\ q$ is in the interior, then we define P'(k) as 1. P'(k) is an extension of P(k) to all positive integers, which contradicts our assumption that P is unextendible.

6. Remark

We have proved that it is impossible always to determine whether a rational point is in the interior or on the boundary of a closed set, even if the group is the closed interval with the endpoints that are constructive real numbers. The productive natural line is a particular example of the general concept of constructive topological spaces. In such spaces, we could ask whether we could algorithmically decide whether a point lies in the interior or on the boundary of a closed set. Because of our theorem, we conclude that this problem is also generally undecidable.

7. Conclusion

In a nutshell, we yield the final result that It is generally impossible to determine whether a rational point lies in the interior or on the boundary of a closed set under the constructive mathematics insight. At the same time, this essay could be somewhat improved to be a more compelling one. Specifically, we argue by providing the most straightforward scenario. Namely, the set we took into consideration is 1- dimensional. However, It seems more persuasive and informative if locations in higher dimensional space could be considered. Based on our

IJFANS International Journal of Food And Nutritional Sciences
Official Publication of International Association of Food and Nutrition Scientists

research outcomes, some results in classic calculus need to be somewhat modified. For instance, In traditional calculus, if we are assigned a task to find the extreme value of a derivable function, we scheme to glean all the points where the derivative becomes zero and compare the value of the process at these points and the boundary points. Howbeit, If we are willing to encode a computer program to carry out this task, this scheme needs to be altered. The reason is that even though we know such an extreme point exists, the computer may be unable to tell us where it is obtained under some circumstances since it needs infinite time to find the exact value.

References

[1] Bishop, E. Beeson, M. (2013). Foundations of constructive analysis. Ishi Press International.

[2] Mints, G. E.: 1983, 'Stepwise Semantics of A. A. Markov' (Russian), a supplement to the Russian translation of the Handbook of Mathematical Logic, Nauka, Moscow, part IV, pp. 348-357.

[3] Shanin N.A., “A hierarchy of ways of understanding judgments in constructive mathematics”, Problems of the constructive direction in mathematics. Part 6, Trudy Mat. Inst. Steklov., 129, 1973, 203–266; Proc. Steklov Inst. Math., 129 (1973), 209–271.

[4] Kushner, B. A. (1985). Lectures on constructive mathematical analysis. American Mathematical Society.

[5] Mandelkern, M. “Constructive Mathematics.” Mathematics Magazine, vol. 58, no. 5, 1985, pp. 272–80.

[6] Turing, A. M. (1936). On Computable Numbers, with An Application to the Entscheidungs problem. Proceedings of the London Mathematical Society, 42, 1936.

[7] Shen, A., and Vereshchagin N.K. Computable Functions. AMS Press, 2003.