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A Note on ψ -Stability Solutions For Semi-Linear Differential Equations on R

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Abstract

The main object of this paper is develop the sufficient conditions for the uniqueness and existence of at least one ψ -stability solution for the semi linear differential equation $x' = A(t)x + b(t) + f(t,x)$ on R with the help of standard principles Schuader Tychnoff theorem. and Banach contraction principle

Key words : ψ -stability, existence and uniqueness, Banach contraction principle, Schuader Tychnoff theorem, semi linear differential equation.

Introduction

The aim of this paper is to give the sufficient conditions for the semi linear differential equation $x' = A(t)x + b(t) + f(t,x)$ ------(1.1)

has at least one ψ -stability solution on R, where $A \in R^{n\times n}$, $f \in R \times R^n$ and $f(t,0) = 0$, and $b(t)$ is any stability, continuous function. Here ψ is a continuous matrix function on R.

The main concept under consideration is to find the sufficient conditions for the existence of a solution under some specified stability ness conditions. A fundamental result of this type, for system of differential equations is studied by Coppel [5, Theorem 4, Chapter V].

The problem of ψ-stability solutions for systems of ordinary differential equations has been studied by many authors like[1– 4,7–10, 12, 13]. The extension of the concept of ψ -stability solutions to Lyapunov matrix differential equations was studied in [11, 14]. The function ψ is consider as a scalar continuous function In the papers [5, 6], and it is a continuous matrix function. The introduction of the matrix function in papers [3, 6 – 11, 14, 15] . ψ permits to obtain a mixed asymptotic behavior of the components of the solutions.

The existence of ψ-stability solutions for semi-linear differential equations on R was not yet discussed. So, here we develop the sufficient condition for existence of ψ-stability solutions for semi liner differential equation **(1.1)**

2. Preliminaries:

Definition 2.1 [10] Denote R^n the Euclidean n-space. For $x = \{x_1, x_2, x_3, \ldots \ldots x_n\} \in R^n$, we define $\|x\| = max\{|x_1|, |x_2|, \ldots \ldots |x_n|\}$ be the norm of x.

Definition 2.2:[10] For a n×n real matrix $A = (a_{ij})$.

We define the norm $\|A\| = \sup_{\|x\| \le 1} \|Ax\|$. It is well-known that $|A| = \max_{1\le i\le n} \sum_{j=i}^{n} |aij|$.

Definition 2.3 [10] Let $\psi_i : R \to (0,\infty)$, i=1, 2,….n, be continuous functions, we define $\psi = diag[\psi_1, \psi_2, \psi_3, \ldots \psi_n]$,

Then $\psi(t)$ is the invertible matrix function on R.

Definition2.3. [10]A function $\varphi : R \to R^n$ is said to be ψ −bounded on R if $\psi\varphi$ is bounded on R.

By a solution of (1.1), we mean a continuously differentiable function: $R \to R^n$ satisfying the system for all t $\in R$.

Let A be a continuous $n \times n$ real matrix and the associated linear differential system $y' = A(t)y$ (1.2)

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Let Y be the fundamental matrix of (1.2) for which $Y(0) = I_n$ (identity $n \times n$ matrix).

Let the vector space R^n be represented as a direct sum of three subspaces X_-, X_0, X_+ such that a solution $y(t)$ of (1.2) is ψ -stability on R if and only if $y(0) \in X_0$ and ψ-stability on $R_+ = [0, \infty)$ if and only if $y(0) \in X_- \oplus X_0$.

Also let P_{-1}, P_0, P_1 denote the corresponding projection of R^n onto X_-, X_0, X_+ respectively.

In the general case where $(P_0 \neq 0)$, the solution for (1.1) is as follows. $x(t) = \int_{-\infty}^{t} Y(t)P_{-1}Y^{-1}(s)f(s,x(s))ds + \int_0^t Y(t)P_0\, Y^{-1}(s)f(s,x(s))ds - \int_t^{\infty} Y(t)P_1 Y^{-1}(s)f(s,x(s))ds.$

For simplicity assume that the homogeneous equation (1.2) has no non-trivial stability solution $(P_0 = 0)$.

The trivial solution of the vector differential equation x ′ = f(t,x) (where x ∈ R n and f is a continuous n vector function) is said to be Ψ−stable on R+ if for every ε > 0 and every t0 ∈ R+, there exists δ = δ(ε,t0) > 0 such that any solution x(t) of the equation which satisfies the inequality k Ψ(t0)x(t0)k < δ, exists and satisfies the inequality k Ψ(t)x(t)k < ε for all t ≥ t_0.

The following two lemmas are useful in proving the main results.

Lemma2.1.[6] Let $Y(t)$ be an invertible matrix which is a continuous function of t on R^+ and let P be a projection. If there exists a continuous function $\varphi : R_+ \to (0,\infty)$ and a positive constant M such that

$$\int_0^{\infty} \varphi(s)|\ \psi(t)Y(t)PY^{-1}(s)\psi^{-1}(s)|\ ds \le M, \quad \text{for all } t \ge 0,$$

and $\int_0^{\infty} \varphi(s)ds = +\infty$, then there exists a constant $N > 0$ such that $|\psi(t)Y(t)P| \le Ne^{-M^{-1}\int_0^t \varphi(s)ds}$, for all $t \ge 0$.

Consequently, $\lim_{t\to\infty} |\ \psi(t)Y(t)P| = 0$.

Lemma2.2.[6] Let $Y(t)$ be an invertible matrix which is a continuous function of t on R^+ and let P be a projection. If there exists a constant $M > 0$ such that $\int_t^{\infty} |\ \psi(t)Y(t)PY^{-1}(s)\psi^{-1}(s)|ds \le M$, for all $t \ge 0$, then for any vector $x_0 \in R^n$ such that $Px_0 \neq 0$, $\lim_{t\to\infty} \sup\ \|\ \psi(t)Y(t)Px_0\| = +\infty$.

3. Main Results

In this session , we develop the proofs for main theorems.

Theorem 3.1. Assume there are supplementary projections P_{-1}, P_1 and K is a positive constant satisfies

$$\int_{-\infty}^{t} |\psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)|ds + \int_t^{\infty} |\psi(t)Y(t)P_1 Y^{-1}(s)\psi^{-1}(s)|ds \le K. \quad (3.1)$$

Let $f(t,x)$ be a continuous function such that

$$\|\psi(t)[f(t,x) - f(t,y)]\| \le \alpha\, \|\psi(t)(x-y)\| \quad \text{------(3.2)}$$

for $-\infty <$ t $< \infty$, $\|\psi x\| \le \rho$, $\|\psi y\| \le \rho$, where $\alpha K < 1$. Then if $b(t)$ is any stability, continuous function such that

$$|b| = sup_{-\infty<t<\infty}\|\psi(t)b(t)\| \le \frac{\rho(1-\alpha k)}{k}, \text{--------(3.3)}$$

then the equation (1.1) has a unique ψ-stability solution $x(t)$ for which $\|\psi x\| \le \rho$.

Proof. By using the lemmas 1.1 and 1.2 the condition (2.1), we got that $|\psi(t)Y(t)P_{-1}\xi|$ is unbounded for $t \le 0$ if $P_{-1}\xi \neq 0$ and stability for $t \ge 0$, and that $|\psi(t)Y(t)P_1\xi|$ is unbounded for $t \ge 0$ if $P_1\xi \neq 0$ and stability for $t \le 0$. Hence (1.2) has only trivial stability solution.

Define $C_\psi = \{x : R \to R^d : x$ is ψ-stability and continuous on R such that $\|\psi x\| \le \rho\}$ and $\|x\|_\psi = \sup_{t\in R}\|\psi(t)x(t)\|$

Clearly this defines a norm on C_ψ and ($C_\psi, \|\cdot\|_\psi$) is a Banach space.

Let T be a function and for any ψ -stability continuous function $x(t)$ defined by

$$Tx(t) = \int_{-\infty}^{t} Y(t)P_{-1}Y^{-1}(s)\{b(s) + f(s,x(s))\}ds - \int_t^{\infty} Y(t)P_1\, Y^{-1}(s)\{b(s) + f(s,x(s))\}ds. \quad (3.4)$$

Take $\|\psi Tx(t)\|$

$$= |\int_{-\infty}^{t} \psi(t)Y(t)\, P_{-1}Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s) + f(s,x(s))\}ds$$

$$- \int_t^{\infty} \psi(t)Y(t)P_1 Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s) + f(s,x(s))\}\, ds|$$

$$\le \left[\int_{-\infty}^{t} |\psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)|\, ds + \int_t^{\infty} |\psi(t)Y(t)P_1 Y^{-1}(s)\psi^{-1}(s)|\, ds\right]\ \|\psi(s)\{b(s) + f(s,x(s))\}\|$$

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$\leq \rho(1-\alpha K) + K\alpha||\psi(s)x(s)||$
$\leq \rho(1-\alpha K) + K\alpha||x||_\psi$
$\leq \rho$

Which gives $Tx(t) \in C_\psi$ and hence $T:. C_\psi \to C_\psi$.

Now, we will derive that T is a contraction mapping on C_ψ.

Take $||\psi(Tx - Ty)||$ =

$$||\int_{-\infty}^{t} \psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,x(s))\}ds$$
$$-\int_{t}^{\infty} \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,x(s)\}ds$$
$$-[\int_{-\infty}^{t} \psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,y(s))\}ds$$
$$-\int_{t}^{\infty} \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,y(s)ds)\}]||$$

$$\leq \left[\int_{-\infty}^{t}|\psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)|ds + \int_{t}^{\infty}|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)|ds\right]||\psi(s)[f(s,x(s)) - f(s,y(s))]||$$
$$\leq K\alpha||x-y||_\psi$$
$$\leq ||x-y||_\psi$$

Hence $||Tx - Ty||_\psi \leq ||x-y||_\psi$

Therefore, we conclude that on C_ψ. T is a contraction mapping Hence by using Banach contraction principle, we conclude that T has a unique fixed-point x(t) on C_ψ . Thus, for which $||\psi x|| \leq \rho$. the semi -linear differential equation (1.1) has a unique fixed point.

Now Conversely assume that , if $x(t)$ is a solution of (1.1) such that $||\psi x|| \leq \rho$ then $y = x - Tx$ is a ψ −stability solution of the linear equation (1.2) , therefore $y = 0$.

This gives the proof.

If (1.1) in Theorem (2.1) is replaced by the stronger condition, then we have the following theorem.

Theorem 3.2. Suppose that

(i) P_{-1}, P_1 are supplementary projections and a positive constant K satiesfies for $-\infty < t < \infty$

$|\psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)| \leq K_1e^{-\alpha(t-s)}$ for $s \leq$ t

$|\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)| \leq K_2e^{-\beta(s-t)}$ for t $\leq s$,

where K_1, K_2, α, β are positive constants.

(ii) For non-linear function $f(t,x(t))$ satisfying

$||\psi(t)f(t,x(t))|| \leq \gamma||\psi(t)x(t)||$,

where $\gamma\left(\frac{K_1}{\alpha}+\frac{K_2}{\beta}\right) < 1$. Then, has at least one ψ-stability solution exists for the nonlinear differential equation (1.1) on R.

proof. We prove this theorem by means of Schuader-Tychnoff fixed point theorem. Let C denote the set of all continuous functions for t ≥ t_o

Let F denote the subset of those continuous functions $x(t)$ for which $||\psi(t)x(t)|| \leq \rho$. Let $T : F \to F$ defined by (2.3).

Result (1). T is continuous, let $\{x_n(t)\} \in F$ be such that $x_n(t) \to x(t)$ uniformly on any finite interval $J \subset R$,

consider $||\psi(Tx_n(t) - Tx(t))||$

$$=||\int_{-\infty}^{t}\psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,x_n(s))\}ds$$
$$-\int_{t}^{\infty}\psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,x_n(s))\}ds$$
$$-[\int_{-\infty}^{t}\psi(t)Y(t)P_{-1}Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,x(s))\}ds$$

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$$-\int_t^\infty \psi(t)Y(t)P_1Y^{-1}(s)\psi^{-1}(s)\psi(s)\{b(s)+f(s,x(s))\}ds]||$$

$$\leq K_1\int_{-\infty}^t e^{-\alpha(t-s)}||\psi(s)[f(s,x_n(s))-f(s,x(s))]||ds + K_2\int_t^\infty e^{-\beta(s-t)}||\psi(s)[f(s,x_n(s)-f(s,x(s))]||ds$$

$$\leq K_1\gamma\int_{-\infty}^t e^{-\alpha(t-s)}||\psi(s)\,[x_n(s)-x(s)]\,||\,ds + K_2\gamma\int_t^\infty e^{-\beta(s-t)}||\,\psi(s)[x_n(s)-x(s)]\,||ds$$

Since $x_n(t)\to x(t)$, it follows that $Tx_n(t)\to Tx(t)$ uniformly on any interval, $J\subset R$.
Result (2). TF is uniformly stability.
Consider $||\psi Tx(t)||\leq$

$$K_1\int_{-\infty}^t e^{-\alpha(t-s)}||\psi(s)\{b(s)+f(s,x(s))\}||ds + K_2\int_t^\infty e^{-\beta(s-t)}||\psi(s)\{b(s)+f(s,x(s))\}||ds$$

$$\leq K_1\gamma\int_{-\infty}^t e^{-\alpha(t-s)}||\,\psi(s)x(s)||\,ds + K_2\gamma\int_t^\infty e^{-\beta(s-t)}||\psi(s)x(s)||\,ds$$

$$\leq \rho\gamma\left(\frac{K_1}{\alpha}+\frac{K_2}{\beta}\right)<\rho$$

Hence the functions in the image set TF are uniformly stability and since $Tx(t)$ is a solution of the non-linear differential equation (1.1), their derivatives are uniformly stability. Thus the functions in TF are equi-continuous and hence all the conditions of Schuader-Tychnoff theorem are satisfied. Hence T has at least on fixed point in F.
Thus the non-linear differential equation (1.1) has at least one ψ −bounded solution on R.
Theorem 3.3. Let Y (n) and Z(n) be fundamental matrices of (1.1) and (1.2). Then,
(i) the trivial solution of (1.2) is Ψ-stable on N if and only if there exists a positive constant M such that $\int_{-\infty}^{\infty}|Z(n)\otimes\psi(n)Y(n)|\leq M$ for all n in R
(ii) the trivial solution of (1.1) is Ψ-uniformly stable on N if and only if there exists a positive constant M such that R
Proof. Suppose that the trivial solution of (1.2) is Ψ-stable on N. From Lemma 2.3, it follows that the trivial solution of the corresponding Kronecker product vector differential equation (2.2) is $I_m\otimes\psi$ stable on N. From Theorem 5.3.1, Lemma 5.2.8 and Lemma 2.1.2, it follows that there exists a positive constant M such that the fundamental matrix W = Z(n)Y (n)
of (2.2) satisfies the condition

$$\int_{-\infty}^{\infty}|I_n\otimes Z(n)\otimes\psi(n)Y(n)|\leq M \text{ for all n in R}$$

$|(I_m\otimes\psi(n))W(n)|=|Z(n)\otimes(\psi(n)Y(n))|\leq M$ for all $n\in N$ ----(3.1)
Conversely, suppose that there exists a positive constant M such that
$|Z(n)\otimes(\psi(n)Y(n))|\leq M$ for all $n\in N$. Again, from Lemma 2.1.2, the fundamental matrix $W(n)=Z(n)\otimes Y(n)$ of the (2.2) satisfies the condition (3.1). From Theorem 5.3.1, it follows that the trivial solution of the equation (2.2) is $I_m\otimes\psi$ -Ψ-stable on N. Thus from Lemma 2.1, the trivial solution of the (1.2) is Ψ-stable on N.
The proof of part (ii) is similar, hence we omit it.

Next, here, we obtain sufficient conditions for Ψ-stability of trivial solution of non-linear matrix differential equation (1.1).

Theorem.3.4.
Suppose that:
i)The linear differential equation (1.2) is Ψ-stable on R
ii) there exists a sequence ϕ: $N\to(0,\infty)$ and a positive constant L such that the fundamental matrices Y(n), Z(n) of (1.2), (1,3) satisfies the condition

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$$\sum_{l=0}^{n} I(n)\phi(l)|(Z(n)Z^{-1}(l+1) \otimes (\psi(n)Y(n)Y^{-1}(l+1)\psi^{-1}(l+1)| \leq L, n \in N$$

(ii) F(n, X(n)) , the non-linear matrix function , satisfies

$|\psi(n+1)F(n, X(n))| \leq \dfrac{\alpha(n)}{m\phi(n)}|\psi(n)X(n)|$ -----(3.2) where nonnegative sequence α(n) satisfies

$\sup\limits_{n \geq n_0} \dfrac{\alpha(n)}{\phi(n)} < \dfrac{1}{L}$ for all n, $n_0 \in N$ *and* $X(n) \in R^{mXm}$ -------(3.3)

Then, the trivial solution of the semi linear differential equation (1.1) is Ψ- stable on R.

Proof. From condition (3.2) and Lemma 2.2, we have

$$\left\|(I_m \otimes \psi(n+1))F(n, \hat{X}(n))\right\| \leq |\psi(n+1)F(n, X(n))|$$

$$\leq \frac{\alpha(n)}{\phi(n)}\left\|(I_m \otimes \psi(n))(F(n, \hat{X}(n))\right\|$$

For all n, n_0 in N and $\hat{X}(n) \in R^{m^2}$.

From Lemma 2.1and Theorem 3.2, it follows that the trivial solution of the corresponding Kronecker product vector differential equation (2.1) is $I_m \otimes \psi$ - stable on N. Thus, from Lemma.2.2, the trivial solution of matrix differential equation (1.1) is Ψ- stable on N.

Results: In this paper, we derived the Ψ- stability solution for semi linear differential equation on R, with the help of Banach contraction principle.

Conclusions: The semi-Linear differential equations plays very important role in more fieds like stochastic process , dynamical systems, Data structures probability and. So, for that reason here we developed the conditions for existence the Ψ- stability solution of nonlinear differential system. This work will be helpful for developing the existence of ψ-bounded solution for semi-linear and semi nonlinear differential system.

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