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Research Paper

SUBORDINATION FOR MULTIVALENT ANALYTIC FUNCTIONS WITH RESPECT TO DZIOK-SRIVASTAVA OPERATOR

[1]Siva M, [2] Hema K
Department of Mathematics
Faculty of Arts and Science
Bharath Institute of Higher Education and Research
Chennai, Tamilnadu
[1]sivamurthy04@gmail.com, [2]khemsre7@gmail.com

Address for Correspondence

[1]Siva M, [2] Hema K
Department of Mathematics
Faculty of Arts and Science
Bharath Institute of Higher Education and Research
Chennai, Tamilnadu
[1]sivamurthy04@gmail.com, [2]khemsre7@gmail.com

Abstract

In this article, author studied some inclusion relations and other useful properties of sub-classes of multivalent analytic functions defined by Dziok- Srivastava Operator. In particular S_0^1 (2, 0, 0) coincides with Goodman's class of uniformly convex functions. The radius of α-spirallike of order β (β<1) and order of starlike are computed. A special member of $S_m^1(\alpha_1, \alpha, \mu)$ are also obtained.

Keywords: Dziok- Srivastava Operator, Uniformly convex functions, α-spiral like, Multivalent analytic functions.

Mathematics Subject Classification: 30C45

1. Introduction

Geometric Function Theory which is one of the branches of Complex Analysis relating to certain studies on interesting results connected with Geometry and Analysis. We propose certain properties of Geometric Function Theory with special applications. Geometric Function Theory deals with several physical properties like defining the magnetic field, electromagnetic field, signal processing and wavelets based on the significant classes of functions with reference to analytic functions.

The solution to the problem which is obtained here, if the corresponding geometric behavior is identified. This could be achieved using the specific tools such as differential subordinates, Coefficient techniques, Convolution techniques or duality techniques with polynomial sums and subordinate chains.

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For instance, the coefficient estimates given by Fejer's lemma (for convex functions). Ozaki conditions (univalency) Kakeya-Enostorm theory (location of zeroes), Vietori's theorem (for starlike functions) and moment theory of continued fractions. To be more precise, a particular result relating to Alexander transform was discussed using differential subordinations, with special reference to sharpness by many researchers including the proposed research problem enclosed with this proposal. In the meantime, a Convolution technique seems to be a challenging tool with sharpness of various other results. The process of unifying these procedures of finding a unique procedure to various sub class has a change in itself with respect to approaches.

For certain results related to the close-to-convex functions it is known that there are five functions (and their rotations) that are starlike univalent in the unit disk with their corresponding Taylor's series coefficient as integers. It is expected that these functions can give the necessary coefficient characterization for the respective classes of close-to-convex functions.

Since all the close-to-convex functions are univalent, the Fejer's lemma and Vietori's theorem may be modified in the case of close-to-convex. Similarly Ozacki condition holds good for the class of starlike for itself. The extremal function for the class of starlike functions and convex functions can be expressed as Gaussian hypergeometric function such as $z_2F_1(1,2-\alpha,1,z)$ and $z_2F_1(1,2-2\alpha,2,z)$ respectively where the parameter 'a' lies in the closed interval $[0,1]$. The behavior of the parameter 'a' describes the order of starlikeness and order of convexity. Thus the study of these classes is closely related to the normalized Gaussian hypergemotirc functions. Hence the properties of sum fractional integral operators which can be represented in terms of hypergeometric functions can be characterized by the parameters a,b,c is $z_2F_1(a,b,c;z)$.

The Earth has many natural resources that make life in the modern world possible. Notably natural resources provide fundamental life support, in the form of both consumptive and public-good services. They provide important beneficial uses to the community. The very first step of my proposal is to relate the suitable algorithm in univalent functions in order to find the exploration of the hidden resources under the surface of the earth.

Koebe initiated the study of theory of analytic univalent functions [24]. One of the major problems in this theory that remained for a long time was the well known Bieberbach conjecture or co efficient problem which had been positively settled by de Branges [7] in 1985. There are several excellent text books which provide the necessary background for the study of this subject [9, 20, 21].

A function f (z) analytic in domain Ω of the complex plane $\mathcal{C}$ is said to be univalent or Schlicht in Ω, if $z_1, z_2 \epsilon\, \Omega$ and $z_1 \neq z_2$ imply that $f\ (z_1) \neq f\ (z_2)$ .The function f (z) is locally univalent at a point $z_0 \epsilon\, \Omega$., if it is univalent in some neighborhood of z_0. The necessary and sufficient condition for local univalence at z_0 is that $f'(z_0) \neq 0$ in Ω.

Without loss of generality, the domain of definition of univalent functions can be restricted to the unit disc $\mathcal{U} = \{z: |z| < 1\}$, in view of the Riemann mapping theorem which asserts that any simply connected domain which is not the whole plane can be mapped onto the unit disc by the univalent transformation.

Let $\mathcal{A}$ be the class of functions of the form given below

$$f\ (z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in the open unit disc $\mathcal{U}$, centered at origin and normalized by the condition $f\ (0) = 0$ and $f'(0) = 1$ Also, let δ be the subclass of $\mathcal{A}$ consisting of univalent function in $\mathcal{U}$.

Definition 1.1 [26]: Let f (z) and g(z) be analytic in $\mathcal{U}$. The function f (z) is subordinate to g(z) in $\mathcal{U}$ and $f\ (z) \prec g(z)$, if there exists an analytic function w(z) in such that $|w(z)| \leq |z|$ and $f\ (z) = g\ (w(z)), z\epsilon\mathcal{U}$.

If g(z) is univalent in $\mathcal{U}$, then

$$f\ (z) \prec g(z) \Leftrightarrow f\ (0) = g(0) \text{ and } (\mathcal{U}) \subset g(\mathcal{U}).$$

Defintion 1.2 [28]: A function $f\epsilon\mathcal{S}$ is said to be convex of order α if $\mathrm{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} \geq \alpha$, for $z\epsilon\,\mathcal{U}$, $0 \leq \alpha < 1$. And if for every > 0 , sufficiently small, there is a point $z_0\ \epsilon\ \mathcal{U}$ for which

$$\mathrm{Re}\left\{1 + \frac{z_0 f''(z_0)}{f'(z_0)}\right\} < \alpha + \epsilon.$$

K(α) denotes the class of convex functions of order α. Furthermore K(0)=0 and $K(\alpha) \subset K$.

Definition 1.3 [9]: A domain Ω of the complex plane is said to be stare like with respect to a point $z_0\epsilon\Omega$, if for any fixed point $z_0\epsilon\Omega$, the line segment joining z_0 to any point $z\ \epsilon\ \Omega$ lies in Ω itself. i.e; $z\ \epsilon\Omega$, $0 \leq t \leq 1$, $tz_0 + (1-t), z\epsilon\Omega$ for $0 \leq t \leq 1$. A function $f\epsilon\mathcal{S}$ is said to be starlike with respect to origin, If it maps $\mathcal{U}$ onto a star like domain with respect to the origin.

The subclass of function □□□ which are star like univalent with respect to the origin is denoted by □*.

Definition 1.4 [23]: A function □(□) □ □ is said to be uniformly starlike in, if □(□) is in □* and has the property that for every circular arc □ contained in □ with center □ also in □ the arc □ (□) is starlike with respect to □(□). □□□ denotes the class of all uniformly starlike functions.

Definition 1.5 [22]: A function □(□) □ □ is said to be uniformly convex in, if □(□) is in □ and has the property that for every circular arc □ contained in □, with center □ also in □ the arc □ (□) is convex. □□□ denotes the class of all uniformly convex functions.

Definition 1.6 [29]: A function F is said to be in the class $□_□$ if F □ $□^*$ and $F(z)$ = z$□'$(□) for some □ □ □□□.

Theorem 1.1 [28]: A necessary and sufficient condition for a function □□□ to be convex univalent □ is that

$$\text{Re}\left\{1+\frac{□□''(□)}{□'(□)}\right\} > 0,\ z□\ □.$$

Theorem 1.2 [27]: If □ (z) = z + $\sum_{□=2}^{\infty} □_□ □^□$ □□, z□ □ then $|□_□| \leq 1$ for are n. Equality being obtained by the function $□_0(□) = \frac{□}{1-□}$.

Theorem 1.3 [25]: A function □□□ maps |□| = □ onto a convex for 0≤ r≤ 2 - $\sqrt{3}$. This result is sharp in the sense that there exist function □□□ which do not map |□| = □ on to a convex curve for $r > 2\text{-}\sqrt{3}$.

Theorem 1.4 [28]: A necessary and sufficient condition for the function □ □ □ to be starlike univalent in □ is that

$$\text{Re}\left\{\frac{□□'(□)}{□(□)}\right\} > 0,\ z□\ □.$$

Theorem 1.5 [10]: If □(□)□ □, z□ □ then $\text{Re}\left\{\frac{□□'(□)}{□(□)}\right\} \geq \frac{1}{2}$, z□ □. That is every convex univalent function is starlike of order $\frac{1}{2}$.

Theorem 1.6 [23]: A function □(□) □ □ of the form □ (z) = z+ $\sum_{□=2}^{\infty} □_□ □^□$ is in □ □□ if and only if

$$\text{Re}\left\{\frac{(□-□)□'(□)}{□(□)-□(□)}\right\} \geq 0,\ (z \neq □,\ (□,□))□\ □\times□.$$

Theorem 1.7 [22]: A function □(□) □ □ of the form □ (z) = z+ $\sum_{□=2}^{\infty} □_□ □^□$ is in □□□ if and only if

$$1+\text{Re}\left\{\frac{(□-□)□''(□)}{□'(□)}\right\} \geq 0,\ \ (□,□)□\ □\times□.$$

Theorem 1.8 [29]: A function □(□) □ □ is in $□_□$ if and only if $\left|\frac{□□'(□)}{□(□)} - 1\right| < \text{Re}\left\{\frac{□□'(□)}{□(□)}\right\}$, □□□ . The function $□(□) = 1 + \frac{2}{□^2}□□□\left(\frac{1+\sqrt{□}}{1-\sqrt{□}}\right)^2$ maps the open disc □ onto the parabolic region Ω = {□: □(□) > |□ − 1| } which lies in the sector $\frac{-□}{4}$ < arg (w) < $\frac{□}{4}$. It is seen that

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□(□) □ $□_□$,

If and only if $\frac{□□'(□)}{□(□)} \prec q(z)$. Also □(□) □ □□□, if and only if $1+\frac{□□''(□)}{□'(□)} \prec q(z)$.

If the function □ (z) □$□_□$ is given, then by the equation (2.1), then we have

$$□_□(□_1, □_2, \ldots, □_□ : □_1, □_2, \ldots, □_□)\ □\ (z) = □^□ + \sum_{□=0}^{\infty} \frac{(□_1)_□ \ldots (□_□)_□}{(□_1)_□ \ldots (□_□)_□} \frac{□_{□+□}}{□!} □^{□+□} (□□□)$$

(1.1)

Recently, several authors obtained many interesting results involving the Dziok-Srivastava linear operator $□_□^{□.□}(□_1)$. ([1], [4] - [9], [11], [13], [14], [16], [19] and [28]). It should also be remarked that the Dziok – Srivastava operator $□_□^{□.□}(□_1)$ is a generalization of many other linear operators considered in investigations [2, 3, 12, 15, 17, 18, 20, 30, 31]. In this article, author makes the simple notation as

$$□_□^{□.□}(□_1) = □_□(□_1, □_2, \ldots, □_□ : □_1, □_2, \ldots, □_□)\ (□ \leqq □ + 1; □\ , □ \in □_0).$$

(1.2)

For □ (z) □$□_□$ we find from equations (1.1) and (1.2) that

$$z(□_□^{□.□}(□_1)□\ (z))' = □_1 □_□^{□.□}(□_1 + 1)□\ (z) + (□ - □_1)□_□^{□.□}(□_1)□\ (z),$$

(1.3)

$$□_□^{1,0}(1)□\ (z) = □\ (z),\ □_□^{1,0}(2)□\ (z) = □□'(□) + (1 - □)□\ (z), \qquad (1.4)$$

and

$$□_□^{1,0}(3)□\ (z) = \frac{1}{2}\{□^2□''(□) + 2(2 - □)□□'(□) + (2 - □)(1 - □)□\ (z)\}$$

(1.5)

Also if we write

$$\breve{□}_□^{□.□}(□_1) = □_□(□_1, □_2, \ldots, □_□ : □_1, □_2, \ldots, □_□)\ (□ \leqq □ + 1; □\ , □ \in □_0). \qquad (1.6)$$

Then it follows from the equations (2.3) and (2.8) that

$$z\left(\breve{□}_□^{□.□}(□_1 + 1)□\ (z)\right)' = □_1 \breve{□}_□^{□.□}(□_1)□\ (z) + (p - □_1)\ \breve{□}_□^{□.□}(□_1 + 1)□\ (z)\ \ (□\ (z)\ □□_□).$$

(1.7)

2. Main Results

2.1 Foremost subordination associations and their applications

In order to prove our main results we need the following lamma:

Lemma 2.1[2]: Let □ be convex univalent function in □ and □(0) = 1, □(□□(□) + □) > 0 (□, □ ∈ □). If □ is analytic in □ and □(0) = 1 Hence

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$$\square(\square) + \frac{\square\square'(\square)}{\square\square(\square) + \square} \prec \square(\square),\ (\square \in \square) \Rightarrow \square(\square) \prec \square(\square)(\square \in \square)$$

Lemma 2.2 [15]: Let $\square$ be convex in $\square$ and $\square \geq 0$. Suppose $\square$ is analytic in $\square$ and $\square(\square(\square)) > 0$ If $\square$ is analytic in $\square$ with $\square(0) = \square(0)$. Hence

$$\square\square^2\square''(\square) + \square(\square)\square(\square) \prec \square(\square) \Rightarrow \square(\square) \prec \square(\square)$$

Theorem 2.1: Let $\square \in \square$. If $\square_{\square}^{\square,\square}\square(\square) \in \square - \square\square(\square)$.

Then $\square_{\square}^{\square+1,\square}\square(\square) \in \square - \square\square(\square)$.

Proof: Let $$\square(\square) = \square \frac{\left(\square_{\square}^{\square,\square}\square(\square)\right)'}{\square_{\square}^{\square,\square}\square(\square)}\ (\square \in \square)$$

where $\square$ is analytic in $\square$ and $\square(0) = 1$. Utilizing equation (1.2), the following is obtained

$$\square(\square) + \square = (\square + 1)\frac{\square_{\square}^{\square+1,\square}\square(\square)}{\square_{\square}^{\square,\square}\square(\square)}$$

Differentiating both sode logarithmically w.r.to $\square$ and multiplying with $\square$, one can attain

$$\square(\square) + \frac{\square\square'(\square)}{\square(\square) + \square} = \frac{\square\left(\square_{\square}^{\square+1,\square}\square(\square)\right)'}{\square_{\square}^{\square+1,\square}\square(\square)}$$

From this argument, we affirm

$$\square(\square) + \frac{\square\square'(\square)}{\square(\square) + \square} \prec \square_{\square,\square}(\square)$$

Using Lemma *2.1* and equation (1.4), $\square_{\square,\square}(\square)$ is univalent and convex in $\square$, also $\square\left(\square_{\square,\square}(\square)\right) > \frac{\square+\square}{\square+1}$.

Theorem 2.2: Suppose Suppose $\square \in \square$. If $\square_{\square}^{\square,\square}\square(\square) \in \square - \square\square\square(\square)$, then $\square_{\square}^{\square+1,\square}\square(\square) \in \square - \square\square\square(\square)$.

Proof: From equations (1.1) and (1.3) and theorem *2.1* one can attain

$$\begin{aligned}\square_{\square}^{\square,\square}\square(\square) \in \square - \square\square\square(\square) \quad &\Leftrightarrow \square\left(\square_{\square}^{\square,\square}\square(\square)\right)' \in \square - \square\square(\square)\\ &\Leftrightarrow \square_{\square}^{\square,\square}\square\square'(\square) \in \square - \square\square(\square)\\ &\Rightarrow \square_{\square}^{\square+1,\square}\square\square'(\square) \in \square - \square\square(\square)\\ &\Leftrightarrow \square_{\square}^{\square+1,\square}\square(\square) \in \square - \square\square\square(\square)\end{aligned}$$

Theorem 3.1: Suppose $\square \in \square$. If $\square_{\square}^{\square,\square}\square \in \square\square\square(\square,\square,\square)$, then $\square_{\square}^{\square+1,\square}\square \in \square\square\square(\square,\square,\square)$.

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Proof: Given

$$□_{□}^{□,□}□(□) \in □□□(□,□,□)$$

$\frac{□\left(□_{□}^{□,□}□(□)\right)'}{□(□)} \prec □_{□,□}(□)$, for certain $□(□) \in □ - □□(□)$

For $□(□)$, $□_{□}^{□,□}□(□) = □(□)$ we attain

$$\frac{□\left(□_{□}^{□,□}□(□)\right)'}{□_{□}^{□,□}□(□)} \prec □_{□,□}(□)$$

Letting

$$□(□) = \frac{□\left(□_{□}^{□+1,□}□(□)\right)'}{□_{□}^{□+1,□}□(□)} \text{ and } □(□) = \frac{□\left(□_{□}^{□+1,□}□(□)\right)'}{□_{□}^{□+1,□}□(□)}$$

Hence $□$, $□$ are analytic in $□$ with $□(0) = □(0) = 1$.

Using theorem 2.1,

$$□_{□}^{□+1,□}□(□) \in □ - □□(□) \text{ with } □(□(□)) > \frac{□ + □}{□ + 1}$$

Also

$$□\left(□_{□}^{□+1,□}□(□)\right)' = \left(□_{□}^{□+1,□}□(□)\right)□(□) \qquad (2.1)$$

Differentiating (2.1) both sides w.r.to *z*. we attain

$$\frac{□\left(□\left(□_{□}^{□+1,□}□(□)\right)'\right)'}{□_{□}^{□+1,□}□(□)} = \frac{□\left(□_{□}^{□+1,□}□(□)\right)'}{□_{□}^{□+1,□}□(□)}□(□) + □□'(□)$$

$$= □(□) \cdot □(□) + □□'(□) \qquad (2.2)$$

Using equation (2.2), on can yield

$$□(□(□)) = \frac{□(□(□) + □)}{|□(□) + □|^2} > 0$$

the above said inequality well pleased the conditions prescribed in Lemma 2.1.
Therefore

$$□(□) \prec □_{□,□}(□)$$

3. Conclusion

The Earth has many natural resources that make life in the modern world possible. Notably natural resources provide fundamental life support, in the form of both consumptive and public-good services. They provide important beneficial uses to the community. The very first

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step of my proposal is to relate the suitable algorithm in univalent functions in order to find the exploration of the hidden resources under the surface of the earth. Thus every coefficient bound has unique relation with natural resources and can be found using wave equations along with Levingston algorithm.

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