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ARF-BROWN TOPOLOGICAL QUANTUM FIELD THEORIES OF PIN$^-$ MANIFOLDS

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ABSTRACT: we study the low-energy behavior of a different model, the Majorana chain. The Arf-Brown invariant is a generalization of the more familiar Arf invariant of a spin surface. The Arf invariant admits three quite different-looking descriptions: one using a quadratic refinement of the intersection pairing; one using a mod 2 index of the spin Dirac operator; and one using KO-theory. This paper presents Arf-Brown Topological quantum Field Theories of Pin$^-$ Manifolds. We apply the Arf-Brown theory for studying the Majorana chain with its time-reversal symmetry. The phase predicted to be associated to this system is an example of a special class of phases called symmetry-protected topological (SPT) phases, which are conjectured to correspond to invertible TFTs in the low-energy ansatz. Specifically, it is believed that the group of 2d fermionic SPT phases with a time-reversal symmetry squaring to 1 is isomorphic to $Z/8$, and that the phase of the Majorana chain is a generator. In the low-energy ansatz, this is related to the $Z/8$ classification of 2d pin− reflection positive invertible TFTs, generated by the Arf-Brown TFT Z_{AB}. We give a few different constructions of the Arf-Brown invariant, which is the partition function of Z_{AB}, then construct ZAB. We investigate this by defining the Majorana chain on a pin− 1-manifold with a triangulation, encoding the pin− structure in additional discrete data. We then compute the space of ground states, and prove that these agree with the state spaces of Z_{AB} .

KEYWORDS: Arf-Brown invariant, Majorana chain, symmetry-protected topological (SPT), TFT.

I. INTRODUCTION

Topological field theories:

We begin by defining topological field theories as mathematically formalized by Atiyah [1], inspired by Segal's approach to conformal field theory:

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Definition:
A symmetric monoidal category is a category C together with data of a bifunctor ⊗: C × C → C, an element 1 ∈ C called the unit, and additional data implementing associativtiy and commutativity of ⊗ and the fact that $1\otimes- \cong -\otimes 1 \cong -$, subject to some coherence conditions [2].

For example, associativity is implemented via an associator, a natural isomorphism

$\alpha: (-\otimes-)\otimes - \xrightarrow{\cong} -\otimes (-\otimes-)$ ... (1),

Which, is required to satisfy the pentagon equation, which guarantees that the different ways to rearrange the parentheses in a fourfold tensor product are coherent. We will not list all of these extra data and conditions; it is common to think of a symmetric monoidal category as "a category C with an associative, commutative tensor product ⊗, a unit 1, and some coherence data and conditions that we will not worry about." But for a complete definition.

Definition:
Let C and D be symmetric monoidal categories. A symmetric monoidal functor Z : C → D is a functor sending 1C 7→ 1D and such that $Z(x, y) \cong Z(x)\otimes Z(y)$.

Again, this is not the whole definition, which asks for more, including a natural isomorphism between $Z(-\otimes-)$ and $Z(-)\otimes Z(-)$ and compatibility of this with the data implementing associativty, commutativity, etc. for C and D. Though, it is common to think of symmetric monoidal functors as "functors sending the unit to the unit and commuting with tensor product."

Extended topological field theory solves this problem by categorifying. Roughly speaking, a (weak) k-category is an algebraic structure like a category, but in which there are 2-morphisms between morphisms, 3-morphisms between 2-morphisms, and so on, up to level k. There are many different ways to make this precise. The idea to use higher categories to generalize the Atiyah-Segal definition of TFT was not introduced in a single paper, but was appearing in the work of many [3]. There is a symmetric monoidal k-category $Bord^{\xi}_{n,n-k}$, called the bordism k-category, whose objects are closed (n − k)-manifolds with ξ-structure, whose morphisms are bordisms with ξ-structure between them, whose 2-morphisms are bordisms between those bordisms, again with ξ-structure, and so on up to degree k: the k-morphisms are diffeomorphism classes of n-dimensional bordisms between bordisms between . . . between bordisms, all equipped with ξ-structure.

Invertible TFTs are special examples of TFTs which are almost, but not quite, trivial.

Definition:
Let (C, ⊗, 1) be a symmetric monoidal k-category and x be an object in C. Then x is invertible if there is $X^{-1} \in C$ such that $X\otimes X^{-1} \cong 1$.

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Definition:
A Picard k-groupoid is a symmetric monoidal k-category which is a k-groupoid (all morphisms in all degrees are invertible under composition) and such that every object is ⊗-invertible. Let C be a (small) symmetric monoidal k-category. We can extract two Picard k-groupoids from C.
• The Picard k-groupoid of units C^{x} is the subcategory of ⊗-invertible objects, composition-invertible 1-morphisms between those objects, composition-invertible 2-morphisms between those 1-morphisms, and so forth.
• The Picard k-groupoid completion $\bar{C}$ is formed from C by formally adding inverses for all objects and morphisms. This has the universal property that if D is a Picard k-groupoid, any map C → D factors uniquely through $\bar{C}$.

In particular, a TFT $Z\colon Bord_{n,n-k}^{\xi} \to C$ is invertible iff it factors through $C^{X} \to C$. The universal property mentioned above implies that the space of invertible TFTs is naturally homotopy-equivalent to the space of symmetric monoidal functors [4].

$$\overline{Bord_{n,n-k}^{\xi}} \to C^{X} \;\ldots (2)$$

If D is a Picard k-groupoid, then the geometric realization of its nerve is an E∞-space (under tensor product) which is group like (all objects in D are ⊗-invertible). Therefore it defines a connective spectrum, which we call the classifying spectrum of D and denote |D|. This is a complete invariant of D, up to equivalence of Picard k-groupoids.

It is believed that the low-energy physics of SPT phases is often described by invertible topological quantum field theories (TFTs), which admit a purely mathematical classification, and that the classification of a given class of SPTs often agrees with the classification of the analogous class of invertible TFTs [5]. At the same time, work on the mathematical theory of invertible TFTs has understood their classification as a problem in stable homotopy theory. Freed-Hopkins uses this to answer the classification problem across a wide range of dimensions and symmetry types. In this paper, we explain this perspective on classifying invertible TFTs and SPT phases in a specific setting, focusing on 2-dimensional theories formulated on manifolds with a pin− structure. Freed-Hopkins shows that the group of deformation classes of 2d invertible pin− TFTs is isomorphic to $Z/8$, and is generated by a TFT Z_{AB} whose partition function is the Arf-Brown invariant of a pin− surface, a generalization of the Arf invariant of a spin surface.

II. PRELIMINARIES

Pin structures are generalizations of spin structures to unoriented vector bundles and manifolds.

Definition:
The pin group Pin(k, S, o) associated to the Clifford algebra which is the kernel of the Clifford norm. The spin group Spin(k, S, o) is the subgroup of Pin(k, S, o) which is even in the grading on the Clifford algebra.

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We are interested in the case where k = R, so that the pin and spin groups are Lie groups. If we specialize to $Cl_{\pm n}(R)$, they're compact Lie groups.

Definition:
Let Pin+ n denote the pin group associated to $Cl_n(R)$, and Pin_n^- denote the pin group associated to $Cl_{-n}(R)$. The corresponding spin groups are canonically isomorphic, so we denote either one by Spinn.

Proposition:
Let $Pin_n^{\pm}$ denote either of Pin_n^+ or Pin_n^-. Then, there are group extensions

$$1 \to Spin_n \to Pin_n^{\pm} \to Z/2 \to 1 \dots (3)$$

$$1 \to Z/2 \to Pin_n^{\pm} \to O_n \to 1 \dots (4)$$

Let ρ: H → G be a homomorphism of Lie groups and π : P → M be a principal G-bundle. Recall that a reduction of the structure group of P to H is data (π^1: Q → M, θ) such that

- π^1 : Q → M is a principal H-bundle, and
- θ : Q ×H G → P is an isomorphism of principal G-bundles, where H acts on G through ρ.

An equivalence of reductions (Q1, θ1) → (Q2, θ2) is a map ψ: Q1 → Q2 intertwining θ1 and θ2.

III. ARF-BROWN INVARIANT OF A PIN− SURFACE

In this section, we give various constructions of the Arf-Brown invariant of a pin− surface: intersection theoretic, index-theoretic, and KO-theoretic.

Intersection-theoretic descriptions of the invariants:

The Arf invariant of a spin surface and the Arf-Brown invariant of a pin− surface are complete bordism invariants defined using intersection theory.

The Arf invariant of a spin surface: Let Σ be a closed oriented surface. If $x, y \in H_1(\Sigma; Z/2)$, then the mod 2 intersection number $I_2(x, y) \in Z/2$ is defined by choosing smooth, transverse representative curves for x and y and computing the number of points mod 2 in their intersection. This does not depend on the choice of representatives and defines a non-degenerate bilinear pairing

$$I_2 : H_1\left(\Sigma; \frac{Z}{2}\right) \otimes H_1\left(\Sigma; \frac{Z}{2}\right) \longrightarrow Z/2 \dots (5)$$

A Z/2-quadratic enhancement of I2 is a quadratic form on $H_1(\Sigma; Z/2)$ whose induced bilinear form is I_2. Explicitly, this is a function

$$q : H_1(\Sigma; Z/2) \longrightarrow Z/2 \dots . (6)$$

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The Arf-Brown invariant of a pin− surface. Now suppose Σ is any closed surface (not necessarily oriented). Then $H_1(\Sigma; Z/2)$ still carries a non-degenerate intersection form I2, although $H_1(\Sigma; Z/2)$ may be odd dimensional, and will not admit a symplectic basis in general. In this case, one must consider the following notion:

A $Z/4$-quadratic enhancement of the intersection form on Σ is a function

$$q: H_1(\Sigma; Z/2) \longrightarrow Z/4 \ldots (7)$$

Index-theoretic description of the invariants:
The Arf(-Brown) invariant of a spin (or pin−) surface admits an alternative description in terms of Dirac operators acting on sections of (s)pinor bundles. In the spin case, the Arf invariant corresponds to the mod 2 index or Atiyah invariant of a spin Riemann surface – the mod 2 dimension of the space of holomorphic sections of a theta-characteristic. In the pin− case, the Arf-Brown invariant may be interpreted as the reduced η-invariant of a twisted Dirac operator as defined and studied by Zhang.

KO-theoretic descriptions of the invariants: Here we explain how the analytic indextheoretic invariants of the previous section may be expressed topologically in terms of pushforwards in (twisted) KO-theory.

IV. ARF-BROWN TFT

The Arf-Brown Theory:

In particular, recall the Arf-Brown invariant

$$AB: \Omega_2^{\text{pin}-} \longrightarrow \mu_8 \subseteq C^X \ldots (8)$$

We denote by

$$Z_{AB}: Bord_2^{\text{pin}-} \longrightarrow sAlg_c \ldots (9)$$

The unique TFT with partition function given by AB.

Proposition:
The pin− Arf-Brown TFT

$$Z_{AB}: Bord_2^{\text{pin}-} \longrightarrow sAlg_c \ldots . (10)$$

Assigns the following invariants:
• To a pin− point, Z_{AB} assigns the first Clifford algebra Cl_1.
• To a bounding pin− circle, Z_{AB} assigns an even line C.
• To a non-bounding pin− circle, Z_{AB} assigns an odd line C.

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Proof:
We use Lemma to compute the values of Z_{AB} on closed manifolds in terms of the action of $\pi_*(S)$ on $\pi_*(MT\text{Pin}-)$.

Recall that the homotopy groups of S (respectively $MT\text{Pin}-$) are given by bordism classes of stably framed manifolds (respectively, pin−-manifolds). As usual, let η denote the generator of π1(S), which is represented by the bordism class of the circle S 1 with its Lie group framing (which induces the non-bounding pin− structure). Thus the operation "multiplication by η" on $\pi_*(MT\text{Pin}-)$ may be understood as direct product with the pin− manifold S^1_{nb}.

We have [KT90b] that the class of S^1_{nb} is a generator of $\pi_1(MT\text{Pin}-)$ ∼= Z/2. The class of $S^1_b \times S^1_{nb}$is the unique element of order 2 in $\pi_2(MT\text{Pin}-)$, and its Arf-Brown invariant is $-1 \in C^X$.

It follows that Z_{AB} takes the non-bounding circle to the unique non-trivial character of π1(S) (which takes the class η represented by S 1 nb to −1). Similarly, Z_{AB} takes the pin− point to the unique non-trivial character of $\pi_2(S)$ (which takes the class η 2 represented by S 1 nb × S^1_{nb} to −1), as required.

V. THE TIME-REVERSAL-INVARIANT MAJORANA CHAIN

A symmetry-protected topological (SPT) phase is a topological phase of matter which is invertible under stacking: after stacking with some other phase, it's equivalent to the trivial phase. Though this isn't a mathematical definition, it tells us that equivalence classes of SPTs should form an abelian group. The computation of this abelian group given a dimension and symmetry type has been the subject of considerable recent research activity at the interface of topology and physics.

To classify SPTs, one generally needs a model for phases of matter and equivalences between them.9 Lattice models are a common choice: roughly speaking, an n-dimensional lattice model is a way of assigning to any closed n-manifold M with a simplicial structure the following data.

• A complex vector space H determined by local combinatorial data on M, called the state space; and
• A self-adjoint operator H: H → H also determined by local combinatorial data, called the Hamiltonian.

Ansatz implies in particular that the group of equivalence classes of d-dimensional SPTs with a given symmetry type is isomorphic to the group of deformation classes of reflection positive invertible (d + 1)-dimensional TFTs with the same symmetry type, a fact which Freed-Hopkins use to classify fermionic SPTs.

Context for the Majorana chain: The Majorana chain is a 2d fermionic SPT phase with time-reversal symmetry making it into a pin− phase, and several physical arguments have shown that it's the generator of the Z/8 of such phases. Ansatz implies that its low-energy field theory is a

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tensor product of an odd number of copies of the Arf-Brown theory. In what follows, we will formulate the Majorana chain on a pin− 1-manifold and study its low-energy behavior.

Defining the Majorana chain:

Let M be a compact pin− 1-manifold with a simplicial structure. Associated to each vertex $v \in \Delta^0(M)$, we associate a trivialized odd line $C_v^{0|1}$ and define the local state space $\mathcal{H}_v := \Lambda(c_v)$. The state space for the Majorana chain on M is

$$\mathcal{H} := \bigotimes_{v \in \Delta^0(M)} \mathcal{H}_v \ldots. (11)$$

Let F denote the space of functions $\Delta^0(M) \longrightarrow C$, regarded as a purely odd vector space. Then $H \cong \Lambda^*(F)$, and hence $\mathcal{H}$ is generated by the δ-functions δv for $v \in \Delta^0(M)$, where each δv is odd.

The low-energy TFT:

We'd like to use Ansatz to determine the deformation class of the low-energy theory Z of the Majorana chain, but it doesn't tell us everything. For example, neither pin⁻ structure on RP^2 is bordant to a disjoint union of mapping tori, so we won't be able to calculate Z(RP^2). Nonetheless, Ansatz tells us we can compute the state space of any closed 1-manifold and the partition functions of all pin− tori and Klein bottles. In particular, we'll find that Z(S^1_{nb}) is an odd line, which is enough to imply that Z is one of the four generators of the Z/8 of deformation classes of reflection positive 2d pin⁻ invertible field theories.

Let π: $M^1 \to$ M be the orientation double cover, and give M^1 the simplicial structure which makes π a simplicial map. The orientation of M induces an orientation of the 0-skeleton of M^1, M_0^1, which is a compact oriented 0-manifold, so this orientation defines a function o: $M_0^1 \to \{\pm 1\}$ sending a positively oriented point to 1 and a negatively oriented point to −1.

$$Let\ n := |\Delta^0(M)| \ldots. (12)$$

Corollary:

Assuming Ansatz, the low-energy TFT Z of the Majorana chain is a generator of the Z/8 of deformation classes of reflection positive pin− invertible field theories. In particular, its deformation class is an odd multiple of the class of the Arf-Brown theory.

Proof. By a result of Schommer-Pries, we know Z is invertible, since there is a pin− structure on S^2 and Z(S_b^1) and Z(S^1_{nb}) are both invertible in sVectC. Since ZAB generates the Z/8 of deformation classes of reflection positive 2d pin− invertible TFTs, Z is deformation equivalent to $(Z_{AB})^{\otimes k}$ for some k, and is a generator iff k is odd. Because Z_{AB} Z(S^1_{nb}) is an odd line, then $(Z_{AB})^{\otimes k}$ (S^1_{nb}) has the same parity as k. Since Z(S_b^1) is odd, then k is odd.

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We can also study the Majorana chain on pin− 1-manifolds with boundary, though again the Hamiltonian depends on an orientation. Kitaev found that the space of ground states on an interval I is twodimensional; from the low-energy perspective, this follows from the fact that for any choice of pin− structure on I, Z(I) is isomorphic to Cl_1as a (Cl_1, Cl_1)-bimodule. We can also see this directly from the lattice.

Thus the ground state is two-dimensional, spanned by a pure tensor whose components are odd for all edges with t(e) = 0 and even otherwise, and a pure tensor whose components are odd for all edges with t(e) = 0 except e∂, and even otherwise. Since Cl_1 is the unique two-dimensional irreducible (ungraded) $Cl_{1,1}$-representation up to isomorphism, the space of ground states on I is isomorphic to either Cl_1 or πCl_1. An argument similar to Proposition shows that we get the former. Finally, to match the left $Cl_{1,1}$-module description of the space of ground states with the (Cl_1, Cl_1)-bimodule description of Z(I), recall that a left Cl_1-action on a module M is equivalent data to a right Cl_1-action on M, which implies the space of ground states on I is Cl_1as a (Cl_1, Cl_1)-bimodule, in accordance with the calculation using the low-energy TFT.

VI. CONCLUSION

In this paper presents Arf-Brown Topological quantum Field Theories of Pin- Manifolds is presented. We applied the Arf-Brown theory for studying the Majorana chain with its time-reversal symmetry. The phase predicted to be associated to this system is an example of a special class of phases called symmetry-protected topological (SPT) phases, which are conjectured to correspond to invertible TFTs in the low-energy ansatz. In the low-energy ansatz, this is related to the $Z/8$ classification of 2d pin− reflection positive invertible TFTs, generated by the Arf-Brown TFT Z_{AB}. We investigate this by defining the Majorana chain on a pin− 1-manifold with a triangulation, encoding the pin− structure in additional discrete data. We then compute the space of ground states, and prove that these agree with the state spaces of Z_{AB}.

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