e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022
Research Paper

A Study on Graph Labeling Problems in Graph Theory

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Abstract

The Graceful Tree Conjecture remains unsolved to these days and there have been a few different approaches researchers have been trying to prove the conjecture. In this section, we present results on the gracefulness of trees and the different ways in which the conjecture has been tackled.

Keywords: Graph Labeling, Problems, Graph theory, applications, mathematics.

Introduction:

As we will discuss in this chapter, paths and caterpillars are graceful. A first approach would be to extend the definition of caterpillars to new families of trees, i.e., look at the class of trees in which the removal of all leaves results in a caterpillar tree - the lobsters-, and so on. However, even the lobster trees have not been characterized yet.

e-ISSN 2320 –7876 www.ijfans.org
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Bermond [4] conjectured in 1979 that all lobsters are graceful. This chapter presents others approaches which have shown to be more interesting.

Conjecture 1.1 (Graceful Tree Conjecture). Every tree is graceful

1.1 Trees with limited diameter

The diameter of a tree T is the maximum distance between two vertices, i.e., diam $(T) = \max\{\text{dist}\ (u, v): u, v \in V(T)\}$. Trees with small diameter have been proved to be graceful. We already showed that trees with diameter 1 (only K_2), diameter 2 (star graphs), and diameter 3 (a subclass of caterpillar trees) are graceful since they are all also caterpillar trees. For greater diameters, Zhao [28] proved in 1989 that all trees with diameter 4 are graceful, Hrnčiar and Haviar [15] proved in 2001 that all trees with diameter 5 are graceful, and Superdock [23,24] proved more recently that some subclasses of trees with diameter 6 are graceful.

We show in this section that all trees with diameter 4 are graceful. The proof presented here was given by Hrnciar and Haviar [15] since it is simpler than the original proof of Zhao [28].

Lemma 1.1. Let T be a tree with a graceful labeling f and let $u \in V(T)$ the verter with $f(u) = 0$. If T^v is the tree obtaincd from T by adding a new vertex v only adjucent to u,then T^v is graceful.

Proof. If m is the number of edges of T, then the vertex labeling f' such that $f'|_{V(T)} = f$ and $f'(v) = m + 1$ is a graceful labeling of T^v.

Corollary 1.1.1. If $w \in V(T)$ has label m, then adding a new vertex only adjacent to w also results in a graceful tree.

Proof. Just consider the complementary graceful labeling of f.

Corollary 1.1.2. If $u \in V(T)$ has label 0(or m) and H is a caterpillar tree, then adding an edge between u and a vertex of H with mani mum eccentricity also results in a graceful tree.

Proof. Apply iteratively Lemma 3.1 giving preference to adding leaves first whenever it is possible. Also note that the corollary is valid for any graceful graph G as long as $u \in V(G)$ has label 0 (or).

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

Lemma 1. 1 allows us to obtain new graceful graphs from smaller ones by adding a vertex. Then, it is reasonable to ask if this could be used to prove the Graceful Tree Conjecture, i.e., somehow show that for any tree, there is a finite sequence of graceful trees starting from a single vertex such that ench tree is the previous one in the sequence plus a vertex, and the last tree of the sequence is the target tree itself.

One sufficient condition to the existence of such sequence is if every tree admits a graceful labeling in which the label 0 can be assigned to any vertex. In the general context, such graphs are called O-rotatable graceful graphs. However, it is not true that every tree is 0-rotatable graceful [26 |.

Let T be a tree and $uv \in E(T)$. We denote by $T_{u,v}$ the subtree of T oontaining v after the removal of the edge uv. Precisely, if $S = \{w \in V(T): v$ is on the uw-path $\}$, then $T_{u,v} = T[S]$.

Lemma 1.2. Let T be a tree with a graoeful labeling f and let $u \in V(T)$ be a verter adjacent to u_1 and u_2. Consider $T^v = T - \left(V\left(T_{u,u_1}\right) \cup V\left(T_{w,u_2}\right)\right)$ and let $v \in V(T^v)$ $v \neq u$ (a) If $u_1 \neq u_2$ and $f(u_1) + f(u_2) = f(u) + f(v)$, then the tree obt ained by a disjoint union of T^v, T_{w,u_1} and T_{u,v_2}, and connecting v to u_1 and u_2 is goaccful with the same graceful labeling f.

(b) If $u_1 = u_2$ and $2f(u_1) = f(u) + f(v)$, then the tree obtained by a disjoint wnion of T and T_{u,w_1}, and connecting v to u_1 is graceful with the same. graceful labeling f.

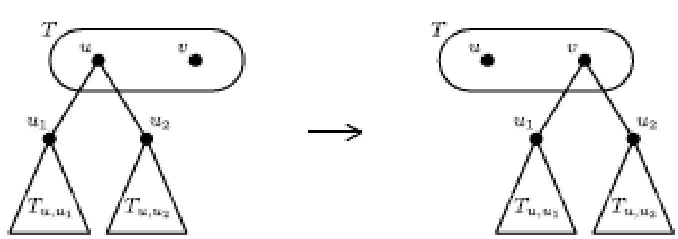
Proof. It suffices to show that the edge labels of uw_1 and uu_2 are the same as of vu_1 and vu_2.

(a)

$$|f(u_1) - f(u)| = |f(u) + f(v) - f(u_2) - f(u)| = |f(v) - f(u_2)|$$
$$|f(u_2) - f(u)| = |f(u) + f(v) - f(u_1) - f(u)| = |f(v) - f(u_1)|$$

(b) $|f(u_1) - f(u)| = \left|\frac{f(u)+f(v)}{2} - f(u)\right| = \left|\frac{f(v)-f(u)}{2}\right|$,

$$|f(u_1) - f(v)| = \left|\frac{f(w) + f(v)}{2} - f(v)\right| = \left|\frac{f(v) - f(v)}{2}\right|$$

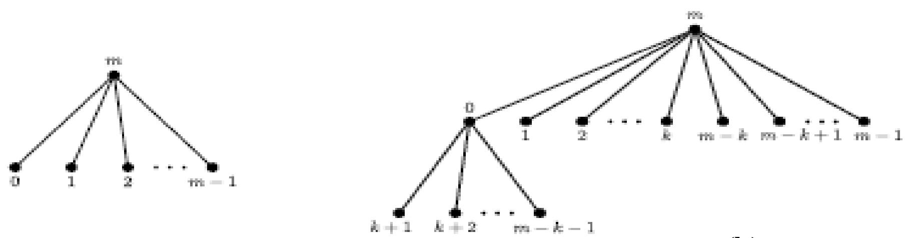


e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

Figure 1.1: Transfer of subtrees from u to v.

This operation is called a transfer and we mostly do transfers of leaves from one vertex to another. For the remaining of this section, for a graceful tree, we no longer distinguish the vertex label from the vertex itself since in a tree every number from $[0, n - 1]$ must appear as a vertex label.

As an example, take the star graph $K_{1,m}$. We can transfer some leaves, which is connected to vertex 0 , to the vertex m (see Figure 3.2). For an example, we can transfer k and $m - k$ from 0 to m since $k + (m - k) = 0 + m$. As said before, the subtree being transferred is usually a leaf and we denote a sequence of transfers of leaves adjacent to u to v as $u \rightarrow v$. Although the notation is not precise, the context will make clear how many and which leaves are being transferred.



(a) (b)

Figure 1.2: Transfer of leaves from m to 0($m \rightarrow 0$ transfer).

Proposition 1.3. All trees with diameter & are graceful

Proof. Consider the following types of transfers.

A $u \rightarrow v$ transfer is of type 1 if the leaves being transferred are $k, k + 1, \ldots, k + s$. This type of transfer can be realized if $u + v = k + (k + s)$. We use this type of transfer when we want to leave an odd number of vertices connected to u.

A $u \rightarrow v$ transfer is of type 2 if the leaves being transferred are $k, k + 1, \ldots, k + s$ and $l, l + 1, \ldots, l + s$ with $k + s < l$. This type of transfer can be realized if $u + v = k + (l + s)$. We use this type of transfer when we want to leave an even number of vertices connected to u.

By Lemma 1.1, it is sufficient to show that every tree T of diameter 4 with central vertex (which is unique in T) of odd degree has a graceful labeling with the central vertex

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

having the maximum label. This is true because, in a tree of diameter 4 , any subtree rooted at one of the children of central vertex is a caterpillar tree.

Let w be the central vertex of T, x be the number of vertices adjacent to w with even degree, and y be the number of vertices adjacent to w with odd degree greater than 1. Let $d(w)=2k+1$ and consider the tree of Figure 3.2b. We can obtain T, from that tree by the following sequence of transfers: $0 \rightarrow m-1 \rightarrow 1 \rightarrow m-2 \rightarrow 2 \rightarrow m-3 \rightarrow \cdots$, where the first x transfers (or $x-1$ if $y=0$) are of type 1 and the next $y-1$ transfers (if $y>1$) are of type 2 .

In order to verify that this sequence works, let us analyse the first transfer. Suppose $\{u_1, \ldots, u_x\}$ is the set of vertices adjacent to w with even degree. Starting with the tree on Figure 3.2 b, the central vertex w is the one with label m. The first transfer is $0 \rightarrow m-1$. Then, u_1 is the vertex 0 and we want to leave $d(u_1)-1$ vertices attached to it. Initially, we have the vertices $k+1, k+2, \ldots, m-k-2, m-k-1$ adjucent to 0 . Since $0+(m-1)=(k+1)+(m-k-2)$, it is possible to leave $d(u_1)-1$ vertices by doing a type 1 transfer of a continuous sequence of vertices to $m-1$. Going on with an analogous analysis, it can be seen that this sequence works.

Proposition 1.4. All trees with diameter 5 are graceful

The proof of Proposition 1.4 also uses the transfers operations used in the proof of Proposition 1.3. However, since it is divided in several cases and it does not add, much to the discussion, we omit it.

Conclusion:

Given that the Graceful Tree Conjecture has remained open for s long time, it is valid to question if it can be false. For that, it would suffice to come up with a tree, that does not admit a graceful labeling. in order to show that a tree does not admit such labeling, one must verify an exponential number of possible ways to label it. Thus, a computational approach is more suited for the task.

Although it is clear that every graph admit an edge-relaxed and a range-relaxed graceful labeling not all graphs have a vertex-relaxed graceful labelling. For thermore, it is still unknown a connected non-graceful graph that has a vertex relaxed graceful labelling.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

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