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Research Paper

A STUDY ON DOMINATION IN GRAPHS

SASIKALA. K
Research Scholar
M.Phil Mathematics
Bharath Institute Of Higher Education And Research
Mail Id : sasikala01061975@gmail.com

Dr. M. KAVITHA
Assistant Professor, Department Of Mathematics
Bharath Institute Of Higher Education And Research

Address for Correspondence

SASIKALA. K
Research Scholar
M.Phil Mathematics
Bharath Institute Of Higher Education And Research
Mail Id : sasikala01061975@gmail.com

Abstract

This thesis entitled “SOME CONTRIBUTIONS TO THE THEORY OF DOMINATION IN GRAPHS” is in the field of graph theory in mathematics. Graphs are basic combinatorial structures. Graphs arose as diagrams representing specific relations among a set of objects. In many disciplines we are faced with situations in which we want to find out how or whether a finite number of objects are related. If the relation is symmetric we can model the situation by a graph. So graph theory as wide applications in the field of computer science, operation research and social science etc.

Introduction

This chapter contains some important basic definitions and results which may be using in the other chapters. For more details, we refer Chartrand and Lesniak[39], Harary [49] and West [83]. Definition 1.1.1. *[14]. A graph is an ordered triple* $G = (V(G), E(G), I_G)$ *where* $V(G)$ *is a nonempty set,* $E(G)$ *is a set disjoint from* $V(G)$ *and* $I(G)$ *is an incidence relation that associates with each element of* $E(G)$*, an unordered pair of elements (same or distinct) of* $V(G)$*. Elements of* $V(G)$ *are called the vertices (or nodes or points) of* G *and* *elements of $E(G)$ are called the edges (or lines) of G. $V(G)$ and $E(G)$ are the vertex set and edge set of G, respectively. If, for the edge e of G, $I_G(e) = u, v$, we write $I_G(e) = uv$.*

Definition 1.1.2. *[14]. If $I_G(e) = \{u, v\}$, then the vertices u and v are called the end vertices or ends of the edge e. Each edge is said to join its ends; in this case, we say that e is incident with each one of its ends. Also, the vertices u and v are then incident with e. A set of two or more edges of a graph G is called a set of multiple or parallel edges if they have the same pair of distinct ends. If e is an edge with end vertices u and v; we write e = uv. An edge for which the two ends are the same is called a loop at the common vertex. A vertex u is a neighbor of v in G, if uv is an edge of G, and u 6= v. The set of all neighbors of v is the open neighborhood of v or the neighbor set of v, and is denoted by $N(v)$; the set $N[v] = N(v) \cup \{v\}$ is the closed neighborhood of v in G. When G needs to be made explicit, these open and closed neighborhoods are denoted by $N(v)$ and $N_G[v]$; respectively. Vertices u and v are adjacent to each other in G if and only if there is an edge of G with u and v as its ends. Two distinct edges e and f are said to be adjacent if and only if they have a common end vertex. A graph is simple if it has no loops and no multiple edges. Thus, for a simple graph G, the incidence function I_G is one-to-one. Hence, an edge of a simple graph is identified with the pair of its ends. A simple graph therefore may be considered as an ordered pair $V(G), E(G)$, where $V(G)$ is a nonempty set and $E(G)$ is a set of unordered pairs of elements of $V(G)$ (each edge of the graph being identified with the pair of its ends).*

Definition 1.1.3. *[14]. A graph is called finite if both $V(G)$ and $E(G)$ are finite. A graph that is not finite is called an infinite graph. Unless otherwise stated, all graphs considered in this thesis are finite.*

Definition 1.1.4. *[14]. A simple graph G is said to be complete if every pair of distinct vertices of G are adjacent in G. Any two complete graphs each on a set of n vertices are isomorphic, each such graph is denoted by K_n.*

Definition 1.1.5. *[14]. A graph is trivial if its vertex set is a singleton and it contains no edges. A graph is bipartite if its vertex set can be partitioned into two nonempty subsets X and Y such that each edge of G has one end in X and the other in Y. The pair (X, Y) is called a bipartition of the bipartite graph. The bipartite graph G with bipartition (X, Y) is denoted by $G(X, Y)$ A simple bipartite graph $G(X, Y)$ is complete if each vertex of X is adjacent to all* *the vertices of Y. If $G(X, Y)$ is complete with $|X| = p$ and $|Y| = q$, then $G(X, Y)$ is denoted by $K_{p,q}$. A complete bipartite graph of the form $K_{1,q}$ is called a star.*

Definition 1.1.6. *[14]. Let G be a simple graph. Then the complement $\bar{G}$ of G is defined by taking $V(\bar{G}) = V(G)$ and making two vertices u and v adjacent in $\bar{G}$ if and only if they are nonadjacent in G. It is clear that $\bar{G}$ is also a simple graph and that $\bar{\bar{G}} = G$.*

Definition 1.1.7. *[14]. A graph H is called a subgraph of G if $V(H) \subseteq V(G)$, $E(H) \subseteq E(G)$ and I_H is the restriction of I_G to $E(H)$. If H is a subgraph of G, then G is said to be a supergraph of H. A subgraph H of a graph G is a proper subgraph of G if either $V(H)$ 6= $V(G)$ or $E(H)$ 6= $E(G)$. A subgraph H of G is said to be an induced subgraph of G if each edge of G having its ends in $V(H)$ is also an edge of H. A subgraph H of G is a spanning subgraph of G if $V(H) = V(G)$. The induced subgraph of G with vertex set $S \subseteq V(G)$ is called the subgraph of G induced by S and is denoted by $G[S]$ or (S).*

Definition 1.1.8. *[14]. A clique of G is a complete subgraph of G. A clique of G is a maximal clique of G if it is not properly contained in another clique of G.*

Definition 1.1.9. *[14]. Let G be a graph and $v \in V$. The number of edges incident at v in G is called the degree of the vertex v in G and is denoted by $d_G(v)$ or simply $d(v)$ when G requires no explicit reference. A loop at v is to be counted twice in computing the degree of v. The minimum (respectively, maximum) of the degrees of the vertices of a graph G is denoted by $\delta(G)$ or δ (respectively, $\Delta(G)$ or Δ). A graph G is called k-regular if every vertex of G has degree k. A graph is said to be regular*

if it is k-regular for some nonnegative integer k. In particular, a 3-regular graph is called a cubic graph.

Energy of a graph

"In this section, we discuss the application of eigenvalues of graphs. The energy of a graph is a concept borrowed from chemistry. Every chemical molecule can be represented by means of its corresponding molecular graph: Each vertex of the graph corresponds to an atom of the molecule, and two vertices of the graph are adjacent if and only if there is a bond

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between the corresponding molecules (the number of bonds being immaterial). The π-electron energy of a conjugated hydrocarbon, as calculated with the Huckel molecular orbital (HMO) method, coincides with the energy (as we are going to define below) of the corresponding graph".[14]

The study of spectral graph theory, in essence, is concerned with the relationships between the algebraic properties of the spectra of certain matrices associated with a graph and the topological properties of that graph. There are various matrices that are associated with a graph, such as the adjacency matrix, the incidence matrix, the Laplacian matrix and the distance matrix. The most common matrix investigated has been the $0 - 1$ adjacency matrix.

The subject had its genesis with the early papers of Lihtenbaum [66] and of Collatz and Sinogowitz [27]. Since that time the subject has steadily grown and has shown surprising interrelationships with other mathematical areas

Domination Related Concepts

Domination theory is one of the important branch of graph theory and the concept of the domination number of a graph was first introduced by Berge [18] in which he introduced the newer topic such as coefficients of external stability,is currently called as the domination number of a graph. Ore [71] published a book on graph theory, in which the words 'dominating set' and 'domination number' are introduced. In 1977, Cockayne and Hedetniemi [25] published the first paper entitled " Optimal domination in graphs". They are the first to used the notation $\gamma(G)$ for the domination number of a graph, which subsequently became the accepted notation. Allan and Laskar [9], Arumugam [12], Sampathkumar [75]and others have contributed significantly to the theory of dominating sets and domination numbers.

Motivated by numerous applications much work has been done on the topic of domination.The problems related to domination is being sought after from past 10 decades, De Jaenisch [31] strived to asertain the minimum number of queens necessary to cover $n \times n$ chess board. Three common kinds of problems that the chess players faced was brought out by Rouse Ball [72] at this time period. These include the following:

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1. covering:To find out the minimum number of chess pieces of given type that is required to attack every square of $n \times n$ chess board.

2. Independent covering: To find out the least number of mutually non attacking chess pieces of a given type that are necessary to dominate every square of $n{\times}n$ board.

3. Independence: Determine the maximum number of chess pieces of a given type that can be placed on $n \times n$ chess board such that no two pieces attack each other. Note that if the chess piece being considered is the queen,this type of problem is commonly known as the$N$-queens problem. Yaglom and Yaglom analysed the above mentioned problem in depth. some of these problems of rooks,knight,kings bishops were solved by them.

An excellent survey on dominating sets and advanced topics of dominating sets can be seen in the two text books authored by Haynes, Hedetniemi and Slater [51, 52]. "A decade later, Cockayne and Hedetniem [25] published a survey paper, in which the notation $\gamma(G)$ was first used for the domination number of G. Since this paper was published, domination in graphs has been studied extensively and several additional research papers have been published on this topic".[76]

An excellent treatment of fundamentals of domination is given in the book edited by Haynes et al [52]

In this section we present some of the basic definitions and theorems which are used in subsequent chapters

Equitable domination in graphs

Realistic inspection makes way for new conception of domination. Nodes with identical ability interact in a positive way within a network. Generally similar ranking people tend to move closely in an organisation.Likewise employees in a company having same authority also make association among themselves.

Balanced distribution of wealth,health,status,etc among citizens is the aim of any democratic country. In the background of this realistic inspection, a graphical representation model is to be createted. The spectrum and the essence of this concept was first recognised by

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Sampathkumar, he went on to introduce diverse varieties of equitability in graphs like Degree equitability, Outward equitability, Inward equitability, equitability in terms of same degree neighbors,or in terms of number of powerful degree neighbors etc. In view of Sampathkumar,if $|\varphi(u) - \varphi(v)| \leq 1$ where $\varphi : V(G) \longrightarrow N$ is a function from the vertex set $V(G)$ into the set of positive integers N, then two vertices $u, v \in V(G)$ are said to be φ -equitable. The birth of the important domination concept called equitable domination originated from the above definition. In [75] the author initiated a study of equitable domination in graphs.“ A subset D of V is called an equitable dominating set if for every $v \in (V - D)$ there exists a vertex $u \in D$ such that $uv \in E(G)$ and $|deg(u) - deg(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by $\gamma_e(G)$ and is called equitable domination number of G”.[10]

Remark: For any graph G, $\gamma(G) \leq \gamma_e(G)$.

“If D is an equitable dominating set then any super set of D is an equitable dominating set.

An equitable set S is said to be a minimal equitable dominating set if no proper subset of S is an equitable dominating set. The minimal upper equitable dominating number is γ_e the upper equitable dominating set of G. If$u \in V$ such that $|deg(u){-}deg(v)| \geq 2$ for every $v \in N(u)$ then u is in every equitable dominating set such points are called equitable isolates. $I_e$denotes the set of all equitable isolates”.[78]

CONNECTED DOMINATION POLYNOMIAL OF GRAPHS

Let $G = (V, E)$ be a simple connected graph of order n. A connected dominating set of G is a set S of vertices of G such that every vertex in $V - S$ is adjacent to some vertex in S and the induced subgraph hSi is connected. The connected domination number $\gamma_c(G)$ is the minimum cardinality of a connected dominating set of G. In this chapter, we introduce the connected domination polynomial of G.
The connected domination polynomial of a connected graph G of order n is the polynomial $D_c(G, x) = \sum_{i=\gamma_c(G)}^{n} d_c(G, i)x^i$, where $d_c(G, i)$ is a number of connected dominating set of G of size i and $\gamma_c(G)$ is a connected domination number of G. We

obtain the connected domination polynomials and connected domination roots of some standards graphs. some interested properties and results are obtained and we characterize the graphs by using the connected domination polynomial.

3.1 Introduction

Throughout this chapter we will consider only a simple connected graphs finite and undirected, without loops and multiple edges. As usual $p = |V|$ and $q = |E|$ denote the number of vertices and edges of a graph G respectively. In general , we use *hxi* to denote the subgraph induced by the set of vertices X. $N(v)$ and $N[v]$ denote the open and closed neighbourhood of a vertex v, respectively. A set D of vertices in a graph G is a dominating set if every vertex in $V - D$ is adjacent to some vertex in D. The domination number $\gamma(G)$ is the minimum cardinality of a dominating set of G. A dominating set S of G is called a connected dominating set if the induced subgraph *hSi* is connected. The minimum cardinality of a connected dominating set of G is called the connected domination number of G and is denoted by $\gamma_c(G)$ [65]. A dominating set with cardinality $\gamma_c(G)$ is called γ_c - set. We denote the family of dominating sets of a graph G with cardinality i by $D_c(G, i)$.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. Then their union $G = G_1 \cup G_2$ is a graph with vertex set $V = V_1 \cup V_2$ and edge set $E = E_1 \cup E_2$. The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets X_1 and X_2 is the graph union $G = G_1 \cup G_2$ together with all the edges joining V_1 and V_2. A spider is a tree T with the property that the removal of all end paths of length 2 of T results in an isolated vertex, called the head of spider. For any real no x , *dxe* denotes the smallest integer greater than or equal to x and *bxc* denotes the largest integer less than or equal to x. The domination polynomial of a graph is introduced by Saeid Alikhani and Yee- hock Peng

POLYNOMIAL OF GRAPHS

There are many ways to represent the graph theory one of the worth way is represent the graph by a polynomial and there are many polynomials represent the graph.

In this chapter, we introduce a new polynomial called the degree polynomial of a graph G. Some properties of the degree polynomial are obtained. Degree polynomials of some standard graphs and of binary graph operations, aside from union, join and corona are obtained.

Finally we have defined the graphical degree polynomial, and the necessary and sufficient conditions for any polynomial $P(x)$ to be graphical degree polynomial are established.

In the last we state some open problems for future work.

4.1 Introduction

Let $G = (V, E)$ be a simple graph of order $|V| = n$. For any vertex $v \in V$ the open neighborhood of v is a set $N(v) = \{u \in V : uv \in E\}$ and the degree of the vertex v is $deg(v) = |N(v)|$, the closed neighborhood set of v is the set $N[v] = N(v) \cup \{v\}$. Let G_1 and G_2 be any two graphs. Then cartesian graph product $G = G_1 \; G_2$ is the graph with vertex set $V_1 \times V_2$ and $u = (u_1, u_2)$ is adjacent to $v = (v_1, v_2)$, whenever $u_1 = v_1$ and u_2 adjacent to v_2 or $u_2 = v_2$ and u_1 adjacent with v_1. By (n, q) - graph we mean simple graph with n vertices and q edges.

Let G_1 be (n_1, m_1) graph and let G_2 be (n_2, m_2) -graph. Then the corona $G_1 \circ G_2$ is defined as the graph G obtained by taking one copy of G_1 and n_1 copies of G_2 and joining the vertex of G_1 to every vertex in the i^{th} copy of G_2. The join graph $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex set V_1 and V_2 is the graph union $G_1 \cup G_2$ together all the edges joining V_1 and V_2.

In this chapter we introduce new type of graph polynomial called degree polynomial of a graph and we obtain some properties of the degree polynomial of a graph. We got the degree polynomial of $G_1 + G_2$, $G_1 \; G_2$, $G_1 \circ G_2$ and some other graphs. Also we define the graphical degree polynomial and we determine when the polynomial $P(x)$ is graphical degree polynomial

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