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A STUDY ON SOME VARIATIONS OF GRAPH

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Abstract:

All the graphs considered here arefinite and undirected. The terms not defined here are used in the sense of Harary [10]. A **graph labeling** is an assignment of integers to the vertices or edges or both subject to certain conditions. If the domain of the mapping is the set of vertices (or edges) then the labeling is called a **vertex labeling (or an edge labeling)**.

By a (p, q) graph G, we mean a graph $G = (V, E)$ with $| V | = p$ and

$| E | = q$.

Graph labeling was first introduced in the late 1960's. Many studies in graph labeling refer to Rosa's research in 1967 [19]. Rosa introduced afunction f from a set of vertices in a graph G to the set of integers $\{0, 1, 2,..., q\}$so that each edge xy is assigned the label $|f(x) - f(y)|$, all labels are distinct. Rosa called this labeling**.** Independently, Golomb [8] studied the same type of labeling and called as **graceful labeling.**

Introduction:

Graceful labeling originated as a means of attacking the conjecture of isomorphic to a given tree with n edges. Those graphs that have some sort of regularity of structure are said to be

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graceful. Sheppard [21] has shown that there are exactly *q*! gracefully labeled graphs with *q* edges. Rosa [19] suggested three possible reasons why a graph fails to be graceful.

1) *G* has “too many vertices” and “not enough edges”

2) *G* has “too many edges”

3) *G* has “wrong parity”

Labeled graphs serve as useful models for a broad range of applications such as X-ray crystallography, radar, coding theory, astronomy, circuit design and communication network addressing. Particularly, interesting applications of graph labeling can be found in [1, 2, 3, 4, 5, 9, 20].

In the recent years, dozens of graph labeling techniques such as

□-labeling, *k* and (*k*, *d*) -graceful, *k*-equitable, skolem graceful, odd graceful and graceful like labeling have been studied in over 1000 papers [7].

In 1990, Antimagic graphs was introduced by Hartsfield and Ringel

[11]. A graph with *q* edges is said to be **antimagic** if its edges can be labeled with 1, 2, …, *q* such that the sums of the labels of the edges incident to each vertex are distinct. This notion of antimagic labeling was extended to hypergraphs by Sonntag [24].

Lee [14] noted that this necessary condition extends to any multigraphwith p vertices and q edges. He conjectured that any connected simple (p,q)

graph with $q(q \square 1) \square \frac{p(p \square 1)}{2} \square \text{mod } p \square$ is edge – graceful and found that this

condition is sufficient for the edge – gracefulness of connected graphs. He also

[15] conjectured that all trees of odd order are edge – graceful.

Small [23] has proved that spiders in which every vertex has odd degreewith the property that the distance from the vertex of degree greater than 2 to each end vertex is the same, are edge – graceful.

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Wilson and Riskin [25] proved that the Cartesian product of any number of odd cycles is edge – graceful.

This thesis emerged with the growing interest in the notion of edge – graceful graphs and its noteworthy conjectures.

By the necessary condition of edge – gracefulness of a graph one can verify that even cycles, and paths of even length are not edge – graceful. But whether trees of odd order are edge – graceful is still open.

Motivated by the notion of edge – graceful graphs and Lo's conjecture, we define a new type of labeling called **strong edge – graceful labeling** by relaxing its range through which we can get strong edge – graceful labeling ofeven order trees.

STRONG EDGE – GRACEFUL LABELINGOF GRAPHS

Introduction

Lo [17] introduced the notion of edge – graceful graphs. The necessary condition for a graph (p, q) to be edge-graceful is $q(q+1) \equiv 0$ or $\frac{p}{2} \pmod p$.

With this condition, even cycles and paths of even length are not edge – graceful. But whether trees of odd order are edge – graceful is still open. On these lines, we define a new type of labeling called strong edge – graceful labeling by relaxing its range in which almost all even order trees are found to be strong edge – graceful.

So, in this chapter, we have introduced the strong edge – graceful labeling of graphs and established the strong edge – gracefulness of fan, twig, path, cycle, star, crown, spider and $P_m \odot nK_1$.

Definition 3.1.1

A graph G with q edges and p vertices is said to have an **edge – graceful labeling (*EGL*)** if there exists a bijection f from the edge set to the set {1, 2, ..., q} so that the induced mapping f^* from the vertex set to the set {0, 1, 2,..., p – 1}

given by $f^*(x) = \left\{\sum f(xy) \mid xy \in E(G)\right\}$ (mod p) is a bijection. A graph G is **edge – graceful graph (*EGG*)** if it admits a edge – graceful labeling

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Definition 3.1.2

A (p, q) graph G is said to have **strong edge – graceful labeling (*SEGL*)** if there exists an injection f from the edge set to $\left\{1, 2, \ldots, \left\lceil \frac{3q}{2} \right\rceil\right\}$ so that the induced mapping f^{+} from the vertex set to $\{0, 1, \ldots, 2p - 1\}$ defined by $f^{+}(x) = \left(\sum f(xy) \mid xy \in E(G)\right)$ (mod $2p$) are distinct. A graph G is said to be **strong edge – graceful graph (*SEGG*)** if it admits a strong edge – gracefullabeling.

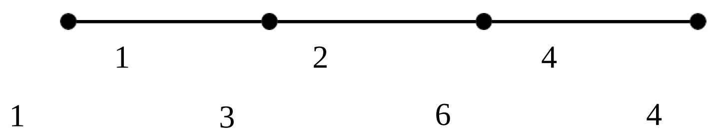
For the clear understanding of the above definitions we illustrate the following examples.

Example 3.1.3

Consider the graph $G = P_4$. Here $q(q + 1) = 3 \times 4 = 12 \equiv 0 \pmod 4$. Thus, Lo's necessary condition is satisfied. But it is not edge – graceful. Because if 3 is assigned to a pendant edge then the labels 1 and 2 cannot be adjacent. Also if 1 is assigned to a pendant edge then the labels 2 and 3 cannotbe adjacent.

Again if 2 is assigned to a pendant edge then either 1 or 3 has to be the label of another pendant edge which is impossible.

But the strong edge – graceful labeling of P_4 is given below.



Thus, P_4 is strong edge – graceful but not edge – graceful.

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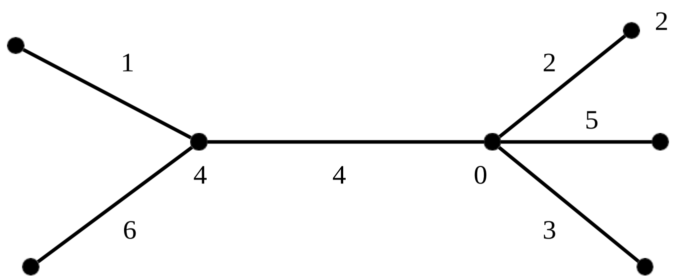
Example 3.1.4

The graph given in this example is both edge graceful and strongedge graceful.

1



Here $p = 7$

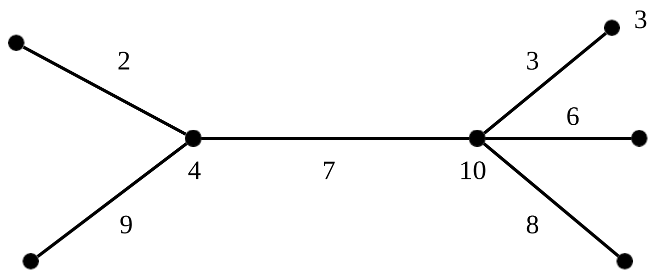
5 $q = 6$

mod $p = 7$

6 3

Fig. 3.1: ***EGL***

2



Here $p = 7$

6 $q = 6$

mod $2p = 14$

9 8

Fig. 3.2: ***SEGL***

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EDGE – GRACEFUL LABELING OF SOME TREES

Introduction

In Chapter – III, we introduced strong edge – graceful labeling and established strong edge – graceful labeling of certain families of graphs. In this chapter, we discuss edge – graceful labeling of some trees such as

- ✦ $K_{1,n} : K_{1,m}$.
- ✦ $N\ K_{1,n} : K_{1,m}$.
- ✦ Y_n.
- ✦ $FC\ (1^m, K_{1,n})$.
- ✦ mG_n.

Edge – graceful labelingDefinition 4.2.1 [17]

A graph $G(V, E)$ is said to be **edge – graceful** if there exists a bijection

f from E to $\{1, 2, ..., q\}$ such that the induced mapping f^* from V to

$\{0, 1, 2, ..., p-1\}$ given by $f^*(x) = \left(\sum f(xy)\right) \bmod p$ taken over all edges

xy is a bijection.

Lo [17] found a necessary condition for a graph with p vertices and

q edges to be edge – graceful as $q(q+1) \equiv 0$ or $\frac{p}{2} \pmod p$.

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and the center v of another star $K_{1,m}$ to a new vertex w. The number of verticesis $n + m + 3$ and the number of edges

Theorem 4.2.3

The graph $K_{1,n} : K_{1,m}$ is edge – graceful for all n, m even and n □ m.

Proof

Let $\{w, v, v', v_1, v_2, ..., v_n, v_{n+1}, ..., v_{n+m}\}$ be the vertices of $K_{1,n} : K_{1,m}$

and $\{e_1, e_2, ..., e_{n+m}\}$ □ $\{e, e'\}$ be the edges of $K_{1,n} : K_{1,m}$ which are denoted as in Fig. 4.1.

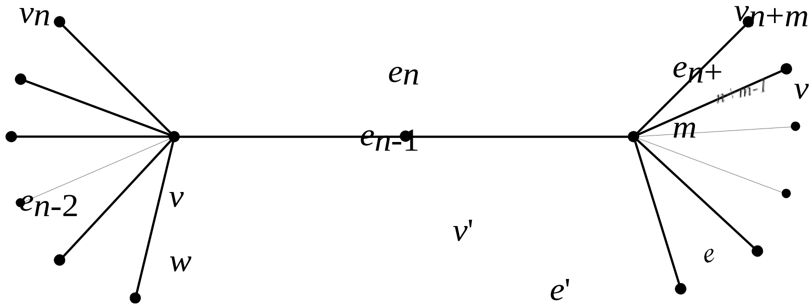

Fig. 4.1: $K_{1,n} : K_{1,m}$ **with ordinary labeling**

We first label the edges as follows:

Consider the Diophantine equation $x_1 + x_2 = p$. The solutions are of the

form $(t, p - t)$ where $1 \square t \square \frac{q}{-}$. There will be2 $\frac{q}{-}$ pair of solutions.2

We first define $f \square e \square = q$; $f \square e' \square = 1$

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SOME MORE STRONG EDGE – GRACEFULGRAPHS

Introduction

In the previous chapters, we have discussed edge – graceful labeling of several families of trees. In this chapter, we present results on the strong edge – gracefulness of some more graphs.

C_m @ $K_{1,n}$, $m \square 3$, $n \square 2$, $m \square n$ when m, n are even.

C_m @ $K_{1,n}$, $m \square 3$, $n \square 2$, $m \square n$ when m and n are odd.

C_m @ $K_{1,n}$, $m \square 3$, $n \square 2$, $m > n$ when m is even and n is odd.

$M(P_n)$, for all $n \square 4$.

$Fl(n)$ ($n \square 4$), when n is even.

$Fl(n)$ ($n \square 4$), when n is odd. for all $t \square 1$ (mod 6).for all $t \square 3$ (mod 6).for all $t \square 5$ (mod 6).for all $t \square 0$ (mod 6).for all $t \square 2$ (mod 6).for all $t \square 4$ (mod 6).

C_n : C_n @ W_k ($n \square 3$, $k \square 2$), for all n and for k is even.

C_m : C_n @ W_2 (m, $n \square 3$), for all $m > n$ where n and m are consecutiveintegers.

C_n : C_n @ W_3 ($n \square 3$), for all n.

C_m : C_n @ W_3 (m, $n \square 3$), for all m, n and $m > n$ where n and m areconsecutive integers.

C_m : C_n @ W_4 ($n \square 3$), for all m, n and $m > n$ where n and m areconsecutive integers.

C_m @ $K_{1,n}$ graphDefinition 5.2.1

The graph C_m @ $K_{1,n}$ is obtained from the cycle C_m by attaching any oneleaf of the star $K_{1,n}$ to any one vertex of the cycle C_m.

Theorem 5.2.2

The graph C_m @ $K_{1,n}$ ($m \square 3$, $n \square 2$) is strong edge – graceful for all

$m \square n$ and m, n are even.

Proof

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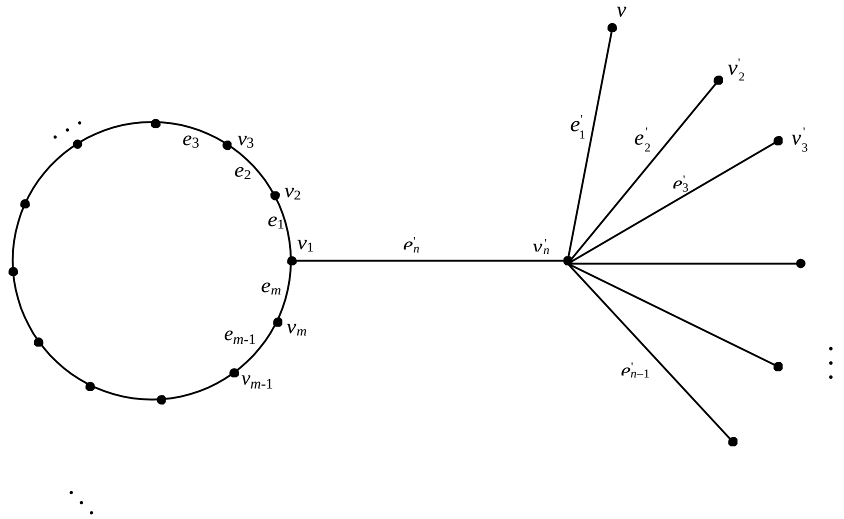
Let $\{v_1, v_2, v_3, ..., v_m, v'_1, v'_2, v'_3, ..., v'_n\}$ be the vertices of C_m @ $K_{1,n}$ and $\{e_1, e_2, e_3, ..., e_m, e'_1, e'_2, e'_3, ..., e'_n\}$ be the edges of C_m @ $K_{1,n}$ which are denoted as in the Fig. 5.1.

' 1



' v

$n-1$

Fig. 5.1: C_m @ $K_{1,n}$ with ordinary labeling

We first label the edges of C_m @ $K_{1,n}$ as follows:

$f(e_i) = i \quad 1 \le i \le m$

$f(e'_i) = m + 2i \quad 1 \le i \le n$

Then the induced vertex labels are:

$f^*(v_i) = 2i - 1 \quad 1 \le i \le m$

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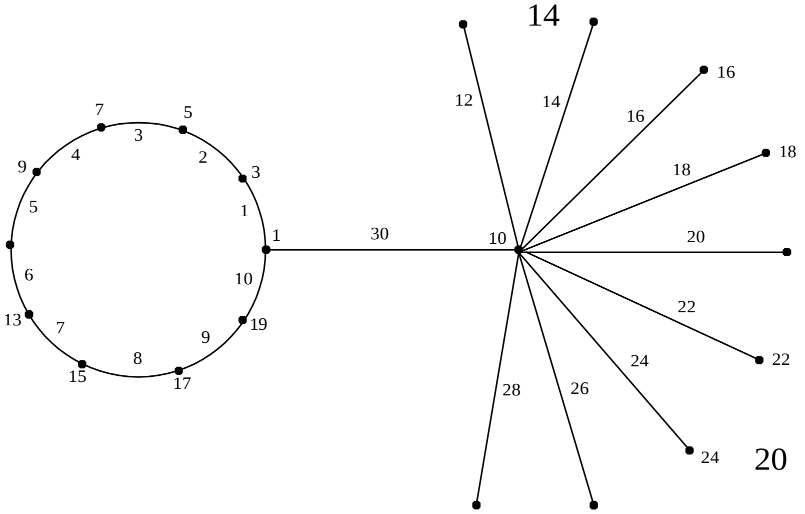
$f^{*}(v'_i) = m + 2i \qquad 1 \le i \le n - 1$

$f^{*}(v'_n) = n$

Clearly, the vertex labels are all distinct. Hence, the graph C_m @ $K_{1,n}$ ($m \ge 3$, $n \ge 2$) is strong edge – graceful for all $m \ge n$ and m, n are even.

The *SEGL* of C_{10} @ $K_{1,10}$ and C_{12} @ $K_{1,6}$ are illustrated in Fig. 5.2 andFig. 5.3 respectively.

12 14

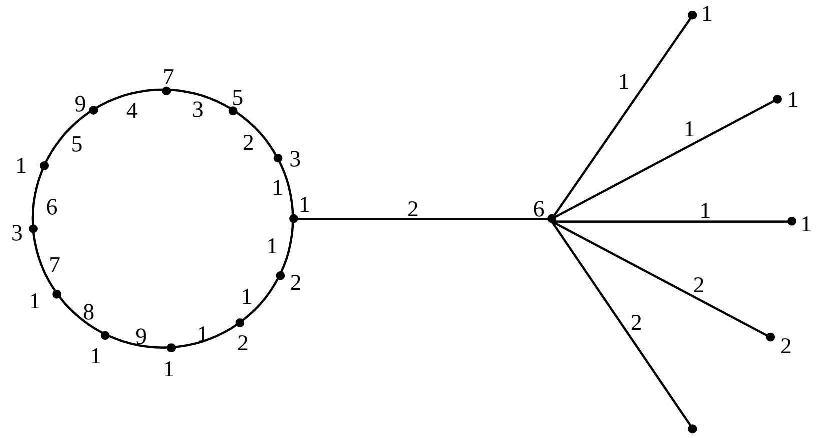


11 20

28 26

Fig. 5.2: C_{10} @ $K_{1,10}$ with *SEGL*

1



8

22

Fig. 5.3: C_{12} @ $K_{1,6}$ **with** ***SEGL***

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BIBLIOGRAPHY

[1] **B.D. Acharya, S. Arumugam, and A. Rosa,** Labeling of Discrete Structures and Applications, Narosa Publishing House, New Delhi, 2008, 1-14.

[2] **G.S. Bloom,** Numbered undirected graphs and their uses**,** A survey of a unifying scientific and engineering concept and its use in developing a theory of non-redundant homometric sets relating to some ambiguities in X-ray Diffraction analysis, Ph. D dissertation, Univ. Southern California, Los Angeles (1975).

[3] **G.S. Bloom and S. W. Golomb,** Numbered complete graphs, unusual rulers, and assorted applications, in Theory and Applications of Graphs, Lecture Notes in Math., 642, Springer-Verlag, New York (1978) 53-65.

[4] **G.S. Bloom and S. W. Golomb,** Applications of numbered undirected graphs, Proc. IEEE, 65 (1977) 562-570.

[5] **G.S. Bloom and D. F. Hsu,** On graceful digraphs and a problem in network addressing, Congr. Numer., 35 (1982) 91-103.

[6] S**. Cabaniss, R. Low, and J. Mitchem**, On edge-graceful regular graphs and trees, Ars Combin., 34 (1992) 129-142.

[7] **J.A. Gallian,** A dynamic survey of graph labeling, The electronic journal of combinatorics 17 (2010) # DS6.

[8] **S.W. Golomb,** How to number a graph, in Graph Theory and Computing, R.C. Read, ed., Academic Press, New York (1972) 23-37.

[9] **R.L. Graham and H.O. Pollak,** On the addressing problem for loop switching, Bell systems Tech. J. 50 (1971) 2495-2519.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

[10] **F. Harary,** Graph Theory – Addison Wesley, Reading Mass (1972).

[11] **N. Hartsfield and G. Ringel,** Pearls in Graph Theory, Academic Press, San Diego,1990.

[12] **J. Keene and A. Simoson**, Balanced strands for asymmetric, edge- graceful spiders, Ars Combin., 42 (1996) 49-64.

[13] **L. M. Lee, S. M. Lee, and G. Murthy**, On edge-graceful labelings of complete graphs-solutions of Lo's conjecture, Congr. Numer., 62 (1988)225-233

[14] **S. M. Lee**, A conjecture on edge-graceful trees, Scientia, 3 (1989) 45- 47.

[15] **S. M. Lee,** New directions in the theory of edge-graceful graphs, Proc. 6^{th} Caribbean Conf. Combin. & Computing (1991) 216-231.

[16] **S. M. Lee, E. Seah and P. C. Wang**, On edge-gracefulness of k^{th} power graphs, Bull. Inst. Math. Acad. Sinica, 18 (1990) 1-11.

[17] **S. Lo**, On edge-graceful labelings of graphs, Congr. Numer., 50 (1985) 231-241.

[18] **G. Ringel**, Problem 25, in Theory of Graphs and its Applications, Proc.Symposium Smolenice 1963, Prague (1964) 162.

[19] **A. Rosa**, On certain valuations of the vertices of a graph, *Theory of Graphs* (International Symposium, Rome July 1966), Gordon and Breach, N.Y. and Dunod Paris (1967) 349-355.

[20] **J. Sedlacek,** Problem 27, in Theory of Graphs and its Applications, Proc. Symposium Smolenice, June, (1963) 163-167.

[21] **D. A. Sheppard**, The factorial representation of major balanced labelled graphs, Discrete Math., 15 (1976) 379-388.

e-ISSN 2320 –7876 www.ijfans.org
Vol.11, Iss.9, Dec 2022

[22] **W. C. Shiu, P. C. B. Lam, and H. L. Cheng**, Edge-gracefulness of the composition of paths with the null graphs, Discrete Math., 253 (2002) 63-76.

[23] **D. Small,** Regular (even) spider graphs are edge-graceful, Congr. Numer., 74 (1990) 247-254.

[24] **M. Sonntag**, Antimagic vertex-labelling of hypergraphs, Discrete Math., 247 (2002) 187-199.

[25] **S. Wilson and A. Riskin**, Edge-graceful labellings of odd cycles and their products, Bulletin of the ICA, 24 (1998) 57-64.