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Research Paper


# A STUDY ON CORDIAL LABELING AND HARMONIOUS LABELING IN GRAPH THEORY

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## Abstract

In this article, author introduces a classes of cycle related called 'One vertex association of substitute shells with a way at any normal level' and it is shown that the graph is elegant, heartfelt and concedes an - $\beta$ valuation, which will help to classify the which are graceful and harmonious.

**Keywords:** cycle related, harmonious graph, graceful graph

## Introduction

Much interest in graph labelings started in mid-1960's with the guess of Ringel (1964) and a paper by Rosa (1967). The famous Ringel-Kotzig (1964, 1965) guess that all trees are smooth, remains disrupted. In his exemplary paper, Rosa (1967) presented - valuation, valuation and other labelings as a device to deteriorate total charts. The -valuation was subsequently called smooth naming by Golomb (1972) and presently this is the term generally broadly utilized. Graham and Sloane (1980) presented agreeable naming regarding their concentrate on added substance bases issue coming from blunder remedying codes. Agile naming and agreeable marking are the two fundamental naming which were broadly contemplated. Varieties of agile and agreeable labelings to be specific, - valuation, exquisite

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marking also, and welcoming naming have been presented with various inspirations in the field of chart naming.

Over the time of forty years, in excess of seven hundred papers have showed up on this subject. This shows the quick development of the field. In any case, the crucial comprehension that the portrayal of smooth and other named graph gives off an impression of being one of the most troublesome and difficult issues in graph hypothesis. Acknowledgment of the least difficult marked chart, specifically, truth be told cheerful graph, is a NP-complete issue, allude Kirchherr (1993). Because of this intrinsic challenges of these labelings, numerous mathematicians have shown interest in giving important circumstances and different adequate circumstances on marked graph wanting to work on the comprehension of the trademark nature of the marked graph. However the area of graph marking fundamentally bargains with hypothetical review, the subject of graph labelings has been the subject of research for quite a while in the applied fields moreover. Noticing that is fascinating the named graph act as helpful models for an expansive scope of utilizations for example, coding hypothesis, X-beam crystallography, radar, stargazing, circuit plan also, correspondence network address ( allude Golomb (1972), Bloom and Golomb (1997) and Bermond (1979)).

A labeling (or valuation) of a graph *G* is an assignment *f* of labels from a set of non-negative integers to the set of vertices of the graph *G* that induce a label for each edge *uv* defined by the labels *f* (*u*) and *f* (*v*). Graceful and harmonious labeling are the two basic labelings which were studied in the field of graph labelings.

**Definition 1.1.1.** A function *f* is called a graceful labeling of a graph *G* with *m* edges, if *f* is an injection from the set of vertices of *G* to the set 0,1,2,...,*m* such that when each edge *uv* is assigned the label *f* (*u*) *f* (*v*) then the resulting edge labels are distinct. A graph which admits graceful labeling is called graceful graph.

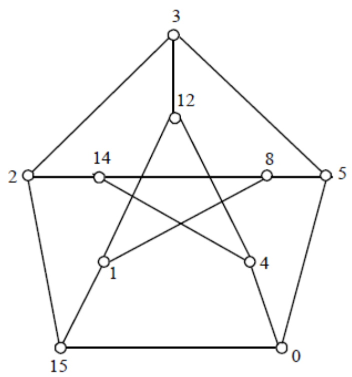

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Figure 1.1: Graceful Labeling of Petersen graph

Harmonious labeling was introduced by Graham and Sloane (1980) in connection with their study on error correcting codes.

**Definition 1.1.2.** A function *f* is called harmonious labeling of a graph *G* with *m* edges, if *f* is an injection from the set of vertices of *G* to the group of integers modulo *m*, such that when each edge *uv* is assigned the label *f* (*u*) *f* (*v*) ( mod *m* ) then the resulting edge labels are distinct. A graph which admits harmonious labeling is called harmonious graph.

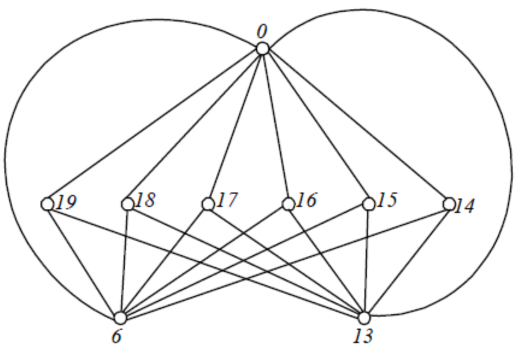


Figure 1.2: Harmonious Labeling of $K_{1, 6, 2}$

Over the period varieties of the effortless and agreeable labelings were presented with various inspirations. $\beta$- valuation is a more grounded form of effortless naming, which was presented by Rosa (1967) in the paper where the effortless naming was first characterized.

**Definition 1.1.3.** A graceful labeling *f* of a graph *G* is called an -valuation of the graph *G* , if there exists an integer such that *f* (*u*) *f* (*v*) (or) *f* (*v*) *f* (*u*) , for every edge *uv* *E*(*G*) , where the integer is called width of the -valuation. Note that a graph which admits $\beta$ -valuation is necessarily a bipartite graph.

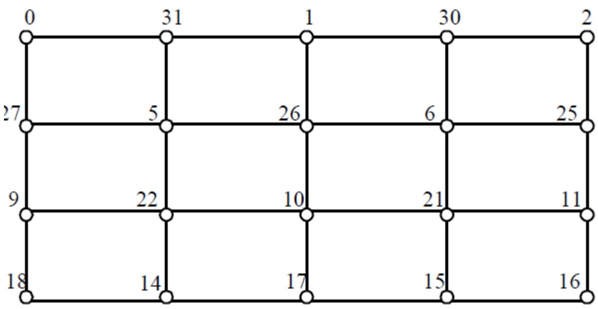


Figure 1.3: $\beta$ − valuation of $P_4$ x $P_5$

Chang (1981) introduced elegant labeling as a variation of harmonious labeling.

**Definition 1.1.4.** A graph *G* with *m* edges is called elegant, if there is an injection *f* :*V* (*G*) 0,1,2,...,*m* such that when each edge *uv* is assigned the label *f* (*u*) *f* (*v*) ( mod *m* 1 ) then the resulting edge labels are distinct and non-zero.

Though the elegant labeling is a slight variation of harmonious labeling, there are graphs which are harmonious but not elegant and vice versa. For example, the cycle $_{4k\ 1}$ *C* is harmonious but not elegant for *k* 1.

On the other hand, the cycle $_{4k}$ *C* is elegant but not harmonious for *k* 1, refer Graham and Sloane (1980) and Chang et al. (1981).

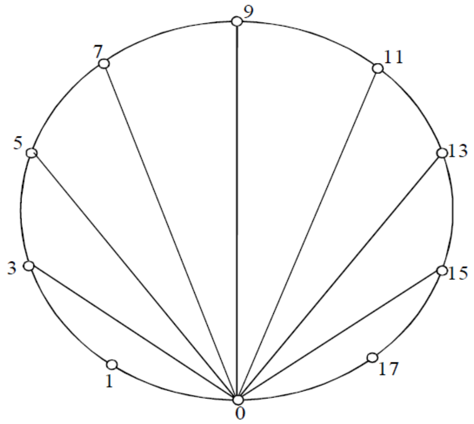


Figure 1.4: Elegant Labeling of $C_{10}$ with 7 chords

It is quite amazing that many intriguing mathematicians have developed bigger agile graphs or other named graphs from specific natural graphs by utilizing different chart activities. Development and Join activities are utilized broadly to such marked graphs.

A latest dynamic survey of graph labeling by Gallian (2005) listed about six hundreds of papers dealing with various graph labelings. Numerous families of graphs are proved to be graceful / harmonious / elegant / cordial and also some of the families are proved to be non-graceful / non-harmonious. But there is no characteristic results on such labeled graphs are achieved. The following observations by Rosa give some general understanding about graceful graphs. Rosa has identified essentially three reasons why a graph fails to be graceful.

## 2.0 Main Results

It is shown that the graph $G(2n_{j,}n_j - 2, k, l_c)$, one vertex union of $k$ alternate shells $C(2n_{j,}\, n_j - 2)$ with a path $P_{2K-1}$ at any common level *l* with chords is graceful and admits an $\beta$ - valuation, for $n_j \geq 3\,, 1 \leq j \leq k, k \geq 1$ and the graph *G*(2*n*, *n* 2, *k*, *l*) is cordial, for $n \geq 3$.

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**Algorithm 2.1**

Stage 1: Let G be a 3-colorable chart.

Stage 2: Allocate 3 tones to a vertex and the entirety of its neighbors if the level of this vertex is bigger than $\sqrt{n}$.

Step 3: There are all things considered $\sqrt{n}$ such vertices and thusly so far at most $3\sqrt{n}$ tones were utilized.

Stage 4: Now, every one of the degrees in the diagram are not exactly $\sqrt{n}$.

Stage 5 : The eager calculation needs all things considered $\sqrt{n}$ tones to shading the rest of the chart.

Step 6: All together, the calculation utilizes $O(\sqrt{n})$ colors.

On the off chance that all precluded vertices are shaded with a similar shading, then, at that point all things considered $2\sqrt{n}+1$ shadings are utilized prior to applying the eager calculation. Consequently, the calculation utilizes about $3\sqrt{n}$ tones.

The degree arrangement of a diagram is limited, non-diminishing grouping of non-negative numbers whose aggregate is even. Conversely, any non-diminishing, positive succession of numbers whose total is even is the degree grouping of some chart.

A hindrance to k-chromaticity (or k-block) is a subgraph that forces each chart that contains it to have chromatic number more noteworthy than k. The complete diagram $K_{(k+1)}$ is a deterrent to k-chromaticity.

**Theorem 2.1.** $\chi(G) \leq \max(1 \leq I \leq n) \min\{d_i + 1, i\}$.

**Proof.** The information request for greedy is $(v_1, v_2 \ldots v_n)$

When shading $v_i$ at most $I - 1$ tones are utilized by its neighbors since greedy has hued just $I - 1$ vertices. When shading $v_i$ at most $d_i$ colors are utilized by its neighbors in light of the fact that the level of $v_i$ is $d_i$.

The info request for greedyis $(v_1, v_2 \ldots v_n)$

When shading $v_i$ at most $d_i'$ colors are utilized by its neighbors.

A marginal improvement to the greedy algorithm.

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An associated non-clique G can be hued with Δ colors where $\Delta \geq 3$ is the most extreme degree in G.

By induction principle, implying the calculation.

Leave v alone a discretionary vertex with degree d(v).

Let G′ = G \ {v}:

On the off chance that G′ isn't an inner circle or a cycle, shading it recursively with Δ colors. If G′ is a coterie, then, at that point a $K_\Delta$ graph can be colored with Δ colors. G′ can't be a $K_{\Delta+1}$) diagram sincethen the neighbors of v would have degree Δ + 1.If G′ is a cycle, then, at that point it ought to be hued with $3 \leq \Delta$ colors.If $d(v) \leq \Delta - 1$, then, at that point shading v with a free tone (categorize argument).If *v* has 2 neighbors hued with a similar shading, then, at that point color v with a free tone (categorize argument). From presently on accept that $d(v) = \Delta$ and that each neighbour of *v* is hued with an alternate tone.

**Remark 2.1** In 1999, Irwing and Manlove [16] presented the idea of b-chromatic number. A b-shading of a chart G is a legitimate vertex shading of G such that each shading class contains a vertex that has something like one neighbor in each and every other shading class and b-chromatic number of a diagram G is the biggest number φ(G) for which Ghas a b-shading with φ(G) colors. A vertex of shading that has any remaining tones in its area is called shading overwhelming vertex. The invariant φ(G) has the chromatic number χ(G) as an inconsequential lower bound, however the contrast between the two of them can be subjective enormous [16]. A unimportant upper destined for φ(G) is Δ(G) + 1.

**Theorem 2.2.** For any $n \geq 3$, $\varphi[P_n \circ_{(n-1)} K_1] = n$.

**Proof.** Leave $P_n$ alone a way diagram of length n-1. That is, $V(P_n) = \{ v_1, v_2, v_3 \ldots v_n\}$ and $E(P_n) = \{e_1, e_2, e_3 .. e_{n-1}\}$. By the meaning of crown item, connect *(n-1)* duplicates of $K_1$ to every vertex of $P_n$. That is, $V[Pn \circ (n\text{-}1) K_1] = \{v_i/1 \leq I \leq n\} \{v_{ij}/1 \leq I \leq n, 1 \leq j \leq n\text{-}1\}$.
$E [Pn \circ (n\text{-}1) K_1] = \{e_i/1 \leq I \leq n\text{-}1\} \{e_{ij}/1 \leq I \leq n, 1 \leq j \leq n\text{-}1\}$.

Consider the shading class C= $\{c_{1,} c_2, c_{3,\ldots} c_n \}$.

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Allocate the shading $c_i$ to vertex $v_i$ for $i = 1,2,3\ldots n$ and relegate the shading $cn+1$ to $v_{ij}$ for $i= 1, 2\ldots n$ and $j = 1, 2, 3\ldots n-1$.

Every $v_i$ is contiguous with $v_i$-1 and $v_i$+1 for $2 \leq I \leq n-1$, $v_1$ is neighboring with $v_2$ and $v_n$ is adjoining with $v_{n-1}$ , because of this non-nearness condition $v_i$ for $1 \leq I \leq n$ doesn't understands its own shading, which doesn't deliver abdominal muscle chromatic shading.

Hence we make the shading as b-chromatic, allot the shading to $v_{ij}$'s as per the following.

For $1 \leq I \leq n$, allocate the color $c_i$ to $v_i$.

For $I = 1, 2, 3\ldots n, j = 1, 2, 3\ldots n-1$, allocate the shading $c_{i+j}$ to $v_{ij}$ when $I + j \leq n$ and appoint the shading $c_{i+j-n}$. At the point when $I + j > n$. Presently the vertices $v_i$ for $I = 1,2,3\ldots n$ understands its own shading which creates a *b*-chromatic shading. Hence forth by shading strategy referenced above said shading is maximal and *b*-chromatic.

**Remark 2.2.** In 1736, Leonhard Euler composed a paper on the Seven Bridges of Konigsberg which is viewed as the primary paper throughout the entire existence of chart hypothesis. Chart hypothesis is a significant apparatus in numerical exploration. A diagram is a theoretical numerical construction shaped by a bunch of vertices and edges joining sets of those vertices. Graphs can be utilized to display the associations between objects, for example, a PC organization can be demonstrated as a chart with every worker addressed by a vertex and the associations between those workers addressed by edges [2,3,11,27].

**Theorem 2.3.** For the line graph $Km,n$, $\varphi[L(K_m,n)] = Max\{m,n\}$ for each $m, n \geq 2$.

**Proof.** Let Km,n be the total bi-partit graph with bipartition (X,Y) where $X = \{v_1,v_2,v_3..vm\}$ and $Y=\{u_1,u_2,u_3..un\}$. Consider the line diagram of Km,n

That is,L (Km,n). Let $v_{ij}$ be the edge between the vertex $v_i$ and $u_j$ for i=1,2,3..m, j=1,2,3..n

That is,$v_i$ $u_j$ = $\{v_{ij}: 1 \leq I \leq m, 1 \leq j \leq n\}$. By the meaning of the line diagram, edges in Km,ncorresponds to the vertices in L(Km,n)

That is,V [L(Km,n)] = $\{v_{ij}: 1 \leq I \leq m, 1 \leq j \leq n\}$. Note that for every I, we say that <$v_{ij}$ : j = 1, 2, 3… n >is a total diagram of request n. Likewise we say for every j, <v ij : i= 1,2,3..m >forms a total diagram of request m. Obviously the quantity of clubs in $L(K_{m,n})$ is m+n.

Case 1: when m < n

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By perception in $L(K_{m,n})$, we have |Kn|>|Km|. Consider the shading class C={$c_1$,$c_2$,$c_3$..cn}. Presently allot an appropriate shading to the vertices as follows. First allocate the shadings to the vertices $v_{ij}$ ($1 \leq I \leq m$, :$1 \leq j \leq n$) as follows.

Here <$v_{mj}$: j=1,2,3..n >for m=1 structures a total chart of request n. Appoint $c_j$ to $v_{1j}$ for j=1,2,3..n, which creates a b-chromatic shading. Assume in the event that we allocate any new shading to the excess completegraph <$v_{ij}$: i= 2 ,3...n, j=1 ,2, 3... n >, it negates the meaning of b-chromatic shading in light of the fact that the leftover complete diagrams doesn't understand the new shading. Therefore to make the shading as b-chromatic one, relegate the shading as follows.

Appoint the shading $c_i$ to the vertex $v_{ij}$when j=1 ,i= 1,2,3..m and allocate cjto $v_{ij}$'swhen i=1, j=1,2,3..n. Next for i=2..3..m and j=2,3… n, appoint the shading $C_{i+j-1}$ to $v_{ij}$'s when $i+j \leq n+1$ and relegate $C_{i+j-(n+1)}$ when i+j> n+1. Presently all the n vertices understand its own shading, which delivers a b-chromatic shading. Along these lines by the shading strategy the above said shading is most extreme and b-chromatic.

Case 2: when m > n

In $L(K_{m,n})$ we have |Km|>|Kn|. Consider the shading class C={$c_1$,$c_2$,$c_3$..$c_n$}. Presently allocate an appropriate shading to the vertices $v_{ij}$ ($1 \leq I \leq m$, $1 \leq j \leq n$)as follows.

Here <$v_{ni}$: i=1,2,3..m>for n=1 structures a total diagram of request n. Appoint cit o vi1for i=1,2,3..m, which creates a b-chromatic shading. Assume on the off chance that we dole out any new shading to the excess complete chart < $v_{ij}$ : i= 2, 3,… m, j = 1, 2, 3...n >,it repudiates the meaning of b-chromatic shading. Therefore to make the shading as b-chromatic one, assignsthe shading to the vertices as follows.

Allot the shading cito $v_{ij}$ when j=1,i=1,2… m and dole out shading $c_j$ to vij'swhen i=1, j=1,2,3..n. Next for i=2,3… ..m and j=2, 3… … n, relegate $C_{i+j-1}$ to vijwhen $i+j \leq m+1$ and allocate $C_{i+j}$-(m+1)when i+j> m+1. Here all m vertices understand its own shading, which

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delivers a b-chromaticcolouring. Hence by the shading system the above said shading is greatest and b-chromatic.Therefore $\varphi[L(K_{m,n})] = m$.

From all the above cases, $\varphi[L(K_{m,n})] = Max\{m,n\}$.

Primary properties of line diagram of complete bipartite graph

Number of vertices in $L(K_{m,n}) = m+n$.

Number of edges in $L(K_{m,n}) = mn/2\ (n+m-2)$.

Greatest degree of $L(K_{m,n})$ is $\Delta = m+n-2$.

Least degree of $L(K_{m,n})$ is $\delta = m+n-2$.

End product:

Each line chart of $K_{m,n}$ is a m+n-2 ordinary diagram.

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