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A Study on Some Contributions in Generalized Cone Graphs

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Abstract:

In this research article, we presented the gracefulness of some graph classes and how to construct bigger graceful graphs from smaller ones. In this chapter, we generalize the wheel graphs, also known as cone graphs, and study its gracefulness. This graph class was first studied by Bhat-Nayak and Selvam [6] in 2003 and not much progress has been made since then.

Keywords: Graphs, Graceful graphs, Generalized cone graphs.

Introduction:

A generalized one graph is the join of a cycle graph C_p and an independent set I_q, where $p \geq 3$ and $q \geq 0$. For instance, for $q = 0$ and $q = 1$, we simply have the cycle graphs and the wheel graphs, respectively.

Throughout this chapter, we denote the vertices of the generalized cone graphs as $V(C_p + I_q) = \{u_0, u_1, \dots, u_{p-1}, v_0, v_1, \dots, v_{q-1}\}$ where $u_k \in V(C_p), u_k u_{k+1} \in E(C_p)$ for

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$0 \leq k < p$ and $u_p = u_0$, and $v_k \in V(I_q)$. Also, from now on, we simply call generalized cone graphs as cone graphs.

The first result we show is concerning the non-graceful cone graphs. As we said in this paper, the only useful theoretical tool for proving the non-existence of graceful labeling for a given graph is the parity condition, which only applies to Eulerian graphs. Thus, supplying the parity condition to Eulerian cone graphs, the following holds.

Proposition 1.1. The cone graph $C_p + I_q$ is not graceful for $p \equiv 2$ (mod 4) and $q \equiv 0(\text{mod}2)$

Proof. For $p \equiv 2(\text{mod}4)$ and $q \equiv 0(\text{mod}2)$, the cone $C_p + I_q$ is Eulerian since the degree of every vertex is even (cf. [7]), and it has $m = p(q + 1)$ edges. Writing $p = 4s + 2$ and $q = 2t$, we have $m = (4s + 2)(2t + 1) \equiv 2$ (mod 4). Hence, by the parity condition, $C_p + I_q$ is not graceful.

1.1 Graceful cones

For $q = 0$ and $q = 1$, we have the cycle graphs and the wheel graphs, respectively, snd their gracefulness is slresdy charscterized in Chapter 2. For $q = 2$, we have the double cones, and it is still an open problem to characterize them. By Proposition 4.1, the double cone $C_p + I_2$ is not graceful for $p \equiv 2$ (mod 4), and so far they are the only non-graceful double cones [6,11,19].

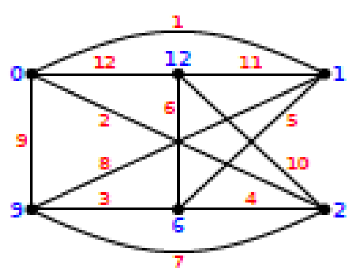

Figure 1.1: Graceful labeling of $C_4 + I_2$.

For the general case, Bhat-Nayak and Selvam [6] proved the following theorem.

Proposition 1.2. The cone graph $C_p + I_q$ is graceful for $p \equiv 0,3$ (mod 12) and $q \geq 1$

For the proof of Proposition 4.2, Bhat-Nayak and Selvam introduced a new graph labeling and showed a more general result similar to Theorem 2.7.

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A vertex labeling f of a graph G with n vertices is ssid to be a special labeling if it satisfies the following conditions:

1 For every $i \in [1, n]$, there exists s vertex $u_1 \in V(G)$ such that $f(u_t)$ is either $2i - 1$ or $2i$.

2 $\mathrm{Im}\,(f_\gamma) = [1,2\pi] \setminus \mathrm{Im}\,(f)$.

3 If $f(x)$ and $f_\gamma(xy)$ are odd, then $f(x) < f(y)$.

Note that conditions 1 and 2 imply that the number of vertices must be the same as the number of edges, i.e., $n = m$.

Theorem 1.3. If a graph G has a special labeling, then the graph $G + I_q$ is graceful for all $q \geq 1$.

Proof. Let G be s graph on p vertices and f be s special labeling of G. Define the vertex labeling g for $G + I_q$ as follows, where $V(G) = \{u_1, \dots, u_p\}$ and $V(I_q) = \{v_1, \dots, v_q\}$

$$g(v_j) = j - 1$$
$$g(u_i) = \begin{cases} i(q+1) & \text{if } f(u_i) = 2i \\ i(q+1) - 1 & \text{if } f(u_i) = 2i - 1 \end{cases}$$

We claim g is a graceful labeling of $G + I_q$. As noted before, since G has a special labeling, G has p edges. Thus, the number of edges of $G + I_q$ is $p + pq$. Clearly, $g\colon V(G + I_q) \to [0, p(q + 1)]$ and it is injective. So, we have to prove that g_γ is onto $[1, p(q + 1)]$. For that, we show that for each $i \in [1, p]$ and $j \in [1, q + 1]$, there is an edge e with $g_\gamma(e) = (i - 1)(q + 1) + j$

Consider s pair (i, j). Since f is a special labeling of G, by condition 1 , there is a vertex $u_i \in V(G)$ with $f(u_i) = 2i - 1$ or $f(u_i) = 2i$.

Case 1. $f(u_i) = 2i - 1$ and $1 \leq j \leq q$.

We have $g(u_i) = i(q + 1) - 1$ and $g(v_{q-j+1}) = q - j$. Since $q - j < i(q + 1) - 1$, the edge label on $u_u v_{q-j+1}$ is $i(q + 1) - 1 - (q - j) = (i - 1)(q + 1) + j$

Case 2. $f(u_1) = 2i - 1$ and $j = q + 1$.

By condition 2 , there is an edge $e = xy \in E(G)$ with $f_\gamma(xy) = 2i$. Hence, $f(x)$, and $f(y)$ have the same parity. Suppose $f(x) = 2a + r$ and $f(y) = 2b + r$, where $r \in \{0,1\}$ is the parity. Then, $f_\gamma(xy) = 2i = |(2a + r) - (2b + r)| = 2|a - b|$, and $i = |a - b|$.

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Therefore, $g_\gamma(xy) = |(a(q+1) - r) - (b(q+1) - r)| = (q+1)|a-b| = i(q+1) = (i-1)(q+1) + (q+1)$

Case 3. $f(u_1) = 2i$ and $2 \leq j \leq q+1$.

We have $g(u_i) = i(q+1)$ and $g(v_{q-j+2}) = q - j + 1$. Since $q - j + 1 < i(q+1)$, the edge label on u_1v_{q-j+2} is $i(q+1) - (q-j+1) = (i-1)(q+1) + j$.

Case 4. $f(u_i) = 2i$ and $j = 1$.

By condition 2 , there is an edge $e = xy \in E(G)$ with $f_\gamma(xy) = 2i - 1$. Now, $f(x)$ and $f(y)$ have different parities. Without loss of generslity, suppose $f(x)$ odd and let $f(x) = 2a - 1$ and $f(y) = 2b$. By condition 3 , we have $f(x) < f(y)$ which implies $g(x) < g(y)$. Thus, $f_\gamma(xy) = 2i - 1 = 2b - (2a - 1)$ implies $i - 1 = b - a$. Finslly, $g_\gamma(xy) = b(q+1) - (a(q+1) - 1) = (b-a)(q+1) - 1 = (i-1)(q+1) - 1$

Thus, we have proved that Im $(g_\gamma) = [1, p(q+1)]$ and therefore g is a graceful labeling of $G + I_q$

We do not present here the complete proof of Proposition 1.2. Here, we only show s partial result which says that $C_{24k} + I_q$ is graceful. For that, Bhat-Nayak and Selvam proved the following lemmas.

Lemma 1.4. For $k \geq 2$, P_{4k-3} has a vertex labeling f such that Im $(f) = [k+2, 2k] \cup [2k+3, 3k+1] \cup [5k+1, 7k-1]$, Im $(f_\gamma) = [2k+1, 6k-4]$, and the end vertices receive the labels $5k+1$ and $7k-1$.

Proof. Let $P_{4k-3} = u_1u_2 \cdots u_{4k-3}$ and define the vertex labeling f as follows:

$$\begin{aligned}
f(u_{2t-1}) &= 5k + i && \text{for } 1 \leq i \leq 2k-1 \\
f(u_{2i}) &= k + 2 && \text{for } i = 1 \\
&= 3k + 3 - i && \text{for } 2 \leq i \leq k \\
&= 3k + 1 - i && \text{for } k+1 \leq i \leq 2k-2
\end{aligned}$$

Now, it is easy to verify directly that Im $(f_\gamma) = [2k+1, 6k-4]$.

Remark 1.1. For $k = 1$, consider the single vertex of P_1 labeled with 6 .

Lemma 1.5. For $k \geq 1$, P_{8k-1} has a vertex labeling f such that Im $(f) = [1, k] \cup [k+2, 8k]$, Im $(f_\gamma) = [1, 8k-2]$, and the end vertioes reccive the labels $2k+1$ and $8k$.

Proof. Let $P_{8k-1} = u_1u_2 \cdots u_{8k-1}$ and define the vertex labeling f as follows:

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$$
\begin{aligned}
f(u_1) &= 2k+1 & & \\
f(u_{2t+1}) &= 4k+1+i & &\text{for } 1 \le i \le k \\
f(u_{2i}) &= 4k+2-i & &\text{for } 1 \le i \le k \\
f(u_{8k+1-2i}) &= 8k+1-i & &\text{for } 1 \le i \le k+2 \\
f(u_{8k-2}) &= 2k+2 & & \\
f(u_{gk-2-2t}) &= i & &\text{for } 1 \le i \le k
\end{aligned}
$$

Thus, we labeled the vertices $u_1, \ldots, u_{2k+1}, u_{6k-3}, \ldots, u\_{Bk-1}$ with labels in $[1, k] \cup [2k+1, 2k+2] \cup [3k+2, 5k+1] \cup [7k-1, 8k]$, and obtained edge labels in $[1,2k] \cup [6k-3, 8k-2]$. For the remaining subpath $u_{2k+1}u_{2k+2} \cdots u_{ek-3}$, label it ss given by Lemma 1.4 to obtain the desired labeling,

Lemma 1.6. For $k \ge 1$, $P_{sk\ -1}$ has a vertex labeling g such that $\mathrm{Im}\ (g) = \{16k+2, 16k+4, \ldots, 18k\} \cup \{18k+4, 18k+6, \ldots, 32k\}$, $\mathrm{Im}\ (f_\gamma) = \{2,4, \ldots, 16k-4\}$, and the end vertioes reccive the labels $20k+2$ and $32k$.

Proof. Let f be the vertex labeling obtained from Lemma 1.5. Then, defining g as $g(u) = 2f(u) + 16k$ gives the required labeling.

Lemma 1.7. For $k \ge 1$, P_{16k+3} has a vertex labeling f such that $\mathrm{Im}\ (f) = \{1,3, \ldots,, 16k-1, 18m+2, 20k+2, 32k, 32k+2, \ldots, 48k\}$, $\mathrm{Im}\ (f_\gamma) = \{16k-2, 16k, 16k+1$ $16k+3, \ldots, 48k-1\}$, and the end vertices receive the labels $20k+2$ and $32k$

Proof. Let $P_{16k+3} = u_1u_2 \cdots u_{16k+3}$ and define the vertex labeling f as follows:

$$
\begin{aligned}
f(u_{2l-1}) &= 20k+2 & &\text{for } i = 1 \\
&= 48k+4-2i & &\text{for } 2 \le i \le 8k+2 \\
f(u_{2i}) &= 2i-1 & &\text{for } 1 \le i \le 7k \\
&= 18k+2 & &\text{for } i = 7k+1 \\
&= 2i-3 & &\text{for } 7k+2 \le i \le 8k+1
\end{aligned}
$$

Now, it is easy to verify that $\mathrm{Im}\ (f_\gamma)$ is as required.

Proposition 1.8. The cone graph $C_{24k} + I_q$ is graceful for all $k \ge 1$.

Proof. Consider P_{8k-1} and P_{16k+3} labeled as given by Lemmas 4.6 and 4.7 respectively.

By joining the paths by identifying the end vertices with the same label, we get & C_{24k} with a vertex labeling f such that $\mathrm{Im}\ (f) = \{1,3, \ldots, 16k-1, 16k+2, 16k+4, \ldots, 48k\}$ and $\mathrm{Im}\ (f_\gamma) = \{2,4, \ldots, 16k, 16k+1, 16k+3, \ldots, 48k-1\}$. Furthermore, the largest odd

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vertex label is less than the smallest even vertex label. Therefore, f satisfies all three conditions of being a special labeling for C_{24k}.

Therefore, by Theorem 1.3, $C_{24k} + I_q$ is graceful.

For the proof of Proposition 1.2, Bhat-Nayak and Selvam proved not only Proposition 1.8, but also that $C_p + I_q$ is graceful for $p \equiv 3,12,15 \ (\mathrm{mod}12)$, each of them following the same strategy as shown before: prowe the existence of a specific vertex labeling of some specific paths and then join their end vertices to form a cycle graph.

Besides Proposition 1.2, Bhat-Nayak and Selvam also proved the following proposition.

Proposition 1.9. The cone graph $C_p + I_q$ is graceful for $p = 7,11,19$ and $q \geq 1$.

Proof. The following vertex labelings are special labelings for their respective cycle.

C_7: 1,14,5,7,10,4,12.
C_{11}: 1,22,5,18,7,15,9,12,14,4,20.
C_{19}: 1,36,3,34,5,32,7,30,12,26,16,22,20,24,13,28,9,17,38.

Brundage [8] also worked on this problem and showed the following result.

Brundage [8] organized the gracefulness of cone graphs in s table (see Table 4.1) and made a conjecture characterizing this class.

Conjecture 1.1 (Brundage, 1994). The generalized cone graph $C_p + I_q$ is graceful if, and only if, the parity condition holds.

Table 1.1: Gracefulness of $C_p + I_q$ (updated x of 2014).

1. 2 Comnputationil results

Questioning the validity of Conjecture 1. 1, we started looking jor counterexample's, i.e., find a cone graph for which the parity condition does nat bold and it is noc graceful For this pask, a backeracking search algurithm similar to the Fany's algorithrn presented in this article was implemented.

The serictery is the same as in Fang's algorithm: it tries to erase a new edge label at each it certain by labeling a not yet labeled vertex. For reducing the search tree, some optimizations were made due to the inherent symmetries of one: graphs. The following observations eliminate mast of search through equivalent labeling given by the symmetries of the graph.

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Conclusion:

we focus on graceful labeling of trees, more specifically, on different ways to approach the Graceful Tree Conjecture. The first one trickle the trees by limiting the diameter by introducing the transfer operation to modify a tree keeping it graceful. The second one reinforces the conjecture by showing computationally that all trees up to 35 vertices are graceful. Finally, we present some relaxed version of graceful labeling in which the better the bound, the closer to the conjecture we showed in this is article , it seems that $C_p + I_q$ is graceful for $p \equiv 0,1,3 \pmod 4$ and $q \geq 1$. For $p \equiv 2 (\mathrm{mod} 4)$, our conjecture says there is a $q_p > 1$ such that the cone graph is not graceful for all $q \geq q_p$. If, moreover, we could find out the parameter q_p for each $p \equiv 2 (\mathrm{mod} 4)$, we would have a characterization of the gracefulness of generalized cone graphs.

Another class of interest is the class of trees, being the main open class on this topic. It is already settled that many classes of trees are graceful, but also there are many classes, even simple ones like lobsters, that are still open. Finally, another approach to the problem is to relax the conditions of graceful labeling and find nearly graceful labeling. This approach by approximating the labeling is also a topic of research for both trees and graphs in general.

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