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Research Paper

APPLICATIONS OF THE THEORY OF HILBERT SPACES WITH RESPECT TO INTEGRAL EQUATIONS

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Abstract

In this article, author tries to describe some applications of the theory of Hilbert spaces to integral equations. The main goal is to illustrate possible applications of techniques developed in theory and to include the standard classification of the important integral equations (Volterra, Fredholm, Integro-Differential, Singular and Abel's integral equations) and their solvability. The most available methods of the subject are abstract and most of them are based on comprehensive theories such as topological methods of functional analysis.

Keywords: Volterra, Fredholm, Integro-Differential, Singular and Abel's integral equations.

Mathematics Subject Classification: 44A15

1. Introduction

Among many contributions to the development of mathematics, the German mathematician David Hilbert (1862 - 1943) is known for his pioneering work in the field of functional analysis [6]. One of the cornerstones of functional analysis, the notion of a Hilbert space, emerged from Hilbert's efforts to generalize the concept of Euclidean space to an infinite dimensional space [7]. The theory of Hilbert space that Hilbert and others developed has not only greatly enriched the world of mathematics but has proven extremely useful in the development of scientific theories, particularly quantum mechanics [1].

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For instance, the ability to treat functions as vectors in a Hilbert space, as permitted by Hilbert space theory, has enabled quantum physicists to solve difficult differential and integral equations by using mere algebra. What is more, the theory and notation of Hilbert space has become so ingrained in the world of quantum mechanics that it is commonly used to describe many interesting phenomenon, including the EPR paradox (entanglement), quantum teleportation, and quantum telecloning [3].

Unfortunately, much of the deep understanding behind Hilbert space theory is often lost in the translation from the mathematical world to the world of physicists. Given the importance of Hilbert space theory to quantum mechanics, a thorough mathematical understanding of the Hilbert space theory that underpins much of quantum mechanics will likely aid in the future development of quantum theory. As such, we explore some of the fundamentals of Hilbert space theory from the perspective of a mathematician and use our insights gained to begin an investigation of one mathematical formulation of quantum mechanics called quantum logic [4].

As we begin our exploration of Hilbert space, the reader is assumed to have some background in linear algebra and real analysis. Nonetheless, for the sake of clarity, we begin with a discussion of three notions that are fundamental to the field of functional analysis, namely metric spaces, normed linear spaces, and inner product spaces [5].

Few definitions are as fundamental to analysis as that of the metric space. In essence, a metric space is simply a collection of objects (e.g. numbers, matrices, pineapple flavored Bon Bons covered with flax seeds) with an associated rule, or function, that determines "distance" between two objects in the space [6]. Such a function is termed a metric. Perhaps the most intuitive example of a metric space is the real number line with the associated metric $|x - y|$, for $x, y \in$ R. In general, though, a metric need only satisfy four basic criteria.

Definition 1.1[7]

A metric space (X, d) is a set X together with an assigned metric function $d : X \times X \to R$ that has the following properties:

Positive: $d(x, y) \geq 0$ for all $x, y, z \in X$, Non degenerate: $d(x, y) = 0$ if and only if $x = y$, Symmetric: $d(x, y) = d(y, x)$ for all $x, y, z \in X$

Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in X$.

Definition 1.2 [8]

A (complex) normed linear space *(L, " • ")* is a linear (vector) space with a function *" • " : L* $\rightarrow$ *R* called a norm that satisfies the properties: Positive: *"v"* ≥ 0 for all v $\in$ L, Non degenerate: *"v"* $= 0$ iff $v = 0$. Multiplicative: *"λv"* = |λ| *"v"* for all *v*∈∈ *L* and λ ∈*C*, Triangle Inequality: *"v + w"* ≤ *"v" + "w"* for all *v, w* ∈ *L*. Recall that a complex conjugate of $a \in C$ is often denoted as *a*. We use this notation throughout this article.

Definition 1.3 [9]

If *V* is a linear space, then a function $(\bullet, \bullet) : V \times V \rightarrow C$ is said to be an inner product provided that Positive: $(v, v) \geq 0$, for all $v \in V$

Non degenerate: $(v, v) = 0$ iff $v = 0$. *1*The study of Hilbert space theory is a subset of the field of functional analysis.

Multiplicative: $(\lambda u, v) = \lambda\ (u, v)$, for all $u, v \in V$ and $\lambda \in C$, Symmetric: $(u, v) = (v, u)$, whenever $u, v \in V$, Distributive: $(u + w, v) = (u, v) + (w, v)$, for all $u, v, w \in V$.

A linear space *V* is defined to be the inner product space $(V, (\bullet, \bullet))$ if it has an inner product defined on it.

The symmetry criterion in Definition 1.3 is sometimes referred to as Hermitian symmetry. For the sake of expediency, a normed linear space *(L, " • ")* is often denoted as *L*. Likewise, an inner product space $(V, (\bullet, \bullet))$ is commonly denoted *V*. In the following two theorems, we formalize our assertions about the relationships between metric spaces, normed linear spaces, and inner product spaces.

2. Methodology

Parallelogram Law [10-11]

The verification of the Parallelogram Law for complex normed direct spaces continues in almost an indistinguishable design as in the genuine case only more chaotic. For this reason, we preclude the more broad evidence of the Parallelogram Law for complex normed straight spaces and rather give the more informational confirmation of the Parallelogram Law for genuine straight vector spaces.

Lemma 2.1. [3,5,7] Leave *L* alone a normed direct space. The standard $k : L \rightarrow R$ is consistent.

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Proof. Let " $L > 0$ and *pick* = ". Presently pick a point an of L and any x 2 L to such an extent that along these lines k. k is consistent on L.

Hypothesis 2.1 (Parallelogram Law).$d(|x|,|a| = |x| - |a| \leq |x - a| < \varepsilon$. and utilize the parallelogram fairness to show that '2 is an inward item space also.

A normed direct space L is an inward item space if and just if its related standard (k k) satises the parallelogram uniformity $|u + v|2 + |u - v|2 = 2|u|2 + 2|v|2$, for each $u; v$ 2 L.

Proof. Assume L is an internal item space with related standard

$$|u + v|2 + |u - v|2 = < u + v, u + v > = < u - v, u - v >$$

$$= < u,u >< u,v >< v,u > + < v,v >< -u,-u > + < u,-v > + < -v,u >$$

$$+ < -v,-v >$$

$$= 2\ |u|\ 2 + 2\ |v|\ 2$$

$$= 2\ |u|2 + 2\ |v|2\ .$$

w 2 L. On the off chance that $u; v$ 2 L,

In this way ‖ • ‖ fulfills the parallelogram equity. Presently guess that L is a normed space whose standard fulfills the parallelogram equity. Characterize a capacity $<.,.>: L \times L \to R$ by $< u, v > = 1,\ |u + v|2 - |u - v|2$, for all $u; v$ 2 L. We wish to show that $h; i$ is an inward item. We start by noticing that $hu; ui = 1$, for all u 2 L.

3. Main Results

Theorem 3.1

A normed linear space $(L, " \cdot ")$ is a metric space with metric d given by $d(v, w) = "v - w"$, where v, w ∈ L.

Proof. That d satisfies the positive and nondegeneracy requirements of metrics, can be seen as an immediate consequence of the positive and nondegeneracy properties of norms. The multiplicative property of normed spaces allows us to make the following simple calculation:

$$d(v, w) = "v - w" = "(-1)(w - v)" = |-1| \cdot "w - v" = "w - v" = d(w, v),$$

which demonstrates that d is symmetric. Finally, we need to show that d satisfies the triangle inequality. To do so, we choose any three $x, y, z \in L$. Since the vectors $x - z$ and $z - y$ are in L, the triangle inequality of norms allows us to see that

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$d(x, y) = "x - y" = "(x - z) + (z - y)" \leq "x - z" + "z - y" = d(x, z) + d(z, y)$

Hence, since d is positive, nondegenerate and symmetric and satisfies the triangle inequality, we conclude that it is indeed a metric.

Theorem 3.2

An inner product space $(V, (\bullet, \bullet))$ is a normed linear space with norm

$"v" = \sqrt{(v, v)}$, for all $v \in V$.

Proof. The positive and non degenerate properties of inner products guarantees that $" \bullet "$ also has these properties.

A simple consequence of Hermitian symmetry $((v, w) = (w, v))$ and the multiplicative property of inner products is the fact that $(v, \lambda w) = \lambda\, (v, w)$ whenever $v, w \in V$ and $\lambda \in C$. Using this equality, we see that

$$"\lambda v" = (\lambda v, \lambda v) = \sqrt{\lambda\lambda}\ (v, v) = |\lambda| \bullet "v".$$

To show that $" \bullet "$ satisfies the triangle inequality criterion of norms, we utilize the Cauchy-Schwartz inequality, which states that if $v, w \in V$, then $|\,(v, w)\,| \leq (v, v)\,(w, w)$.

Since both Karen Saxe [9] and Carol Schumacher [10] provide elegant proofs of the Cauchy-Schwarz inequality, we will use the Cauchy-Schwarz inequality without providing a proof in this article.

Using the distributive property of inner products, we see that for

$$v, w \in V\,,\ "v + w"2 = (v, v) + (w, w) + (v, w) + (w, v)\,.$$

According to the Cauchy-Schwarz inequality then,

$$"v + w"2 \leq "v"2 + "w"2 + "v""w" + "w""v" = ("v" + "w")2\,.$$

So, $"v + w" \leq "v" + "w"$. We conclude that $" \bullet "$ is a norm on $V.Q$

From our exploration of inner product spaces and normed linear spaces, it is useful to pause to examine a couple of interesting examples.

Example 3.1

In this example, we explore the set of all real, bounded sequences, often termed A∞, and show that A∞ is a normed linear space with norm $"(xn)"\infty = sup\{|xn| : n \in N\}$.

Since the absolute value function maps from R to $R+ \cup \{0\}$ (the set of all non-negative real

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numbers), the function *" • "∞* is always positive. Further, if *"(xn)"∞ = 0* for some *(xn)* ∈A∞, then each term in *(xn)* must equal zero, as the least upper bound of the set $\{|xn|: n \in N\}$ is zero. Similarly, if a sequence *(xn)* ∈ *A*∞ is the zero sequence, then, by definition, the supremum of the set *{xn}* must equal zero, which implies that *" • "*∞ is also non-degenerate.

The essential properties of the supremum and outright worth permit us to additional see that for successions *(xn); (yn) 2 '1* and any unpredictable number,

k(xn)k1 = supfj xnj : n 2 Ng = supfj jjxnj : n 2 Ng = j j supfjxnj : n 2 Ng = j j k(xn)k1 ;

what more,

k(xn + yn)k1 = supfjxn + ynj : n 2 Ng supfjxnj + jynj : n 2 Ng supfjxnj : n 2 Ng + supfjymj : m 2 Ng = k(xn)k + k(xn)k ,

Since *k(xn)k1,* satises all models for a standard, '1 is a normed straight space.

Example 3.2

Another significant illustration of a normed direct space is the assortment of all ceaseless capacities on a shut stretch *[a; b]*, indicated *C[a; b]*, with the supremum standard

kfk1 = supfjf(x)j : x 2 [a; b]g:

A comparable to contention to the one given above for '1 shows that *C[a; b*] with standard *kfk1* is for sure a normed straight space.

4. *lp* Spaces

Minkowski's Inequality.

In the past area, we asserted that *l2* is the solitary *lp* space that is an internal item space. Since this is a particularly fascinating property, Prior to doing as such, notwithstanding, we inspect one property that is regular to all *lp* spaces: they are all normed direct spaces. The confirmation of this case depends on Minkowski's Inequality. The technique we use to demonstrate Minkowski's Inequality includes sunken capacities, and depends on the following theorem.

Definition 4.1

A capacity $g : R+ \to R$ is supposed to be inward on the off chance that it fulfills the disparity $\lambda g(x) + (1 - \lambda)g(y) \le g(\lambda x + (1 - \lambda)y)$, for all $x, y \in R+$ and all λ that fulfill $0 \le \lambda \le 1$.

The accompanying figure exhibits a mathematical translation of a sunken capacity.

Figure 1 illustrates a mathematical understanding of an inward capacity. The bend addresses $\lambda g(x)+(1-\lambda)g(y)$ and the line fragment addresses $g(\lambda x)+(1-\lambda)y$.

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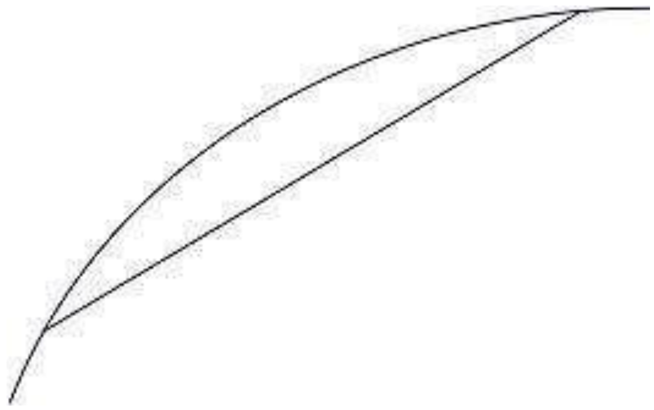

Fig 1

Theorem 4.1 Let *g : R+ → R* be a sunken capacity and characterize a capacity
f : R2→ R by *f (x, y) = y g(x/y)*.
Then, when *Σi=1* we have *f (xi, yi) ≤ f n, i=1*

is substantial for all certain genuine numbers *x1, . . . , xn* and *y1, . . . , yn*.

Proof. We continue by acceptance. Let *λ = y1/(y1 + y2)*. Then, at that point the base case is as per the following:

f (x1, y1) + f (x2, y2) = y1g(x1/y1) + y2g(x2/y2)

= (y + y) y1+g(x/y) + y2g(x/y)

≤ (y1 + y2)g (x1 + x2)/(y1 + y2)

= f (x1 + x2, y1 + y2),

as *1 − λ = y2* . Assume that the imbalance

Σi=1,f (xi, yi) ≤ f k i=1 xi,Σi=1

holds for all certain numbers k stringently not exactly some sure whole number *n*. Then, at that point n

f (xi, yi) = i=1 n−1 f (xi, yi) + f (xn, yn) i=1 n−1 ≤ f i=1

xi, n−1 i=1 yi! + f (xn, yn) ≤ f xn + Σn n−1 i=1 Σ xi, yn + ! n−1 i=1 yi

Hence the proof.

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