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A STUDY OF ARCHEMEDEAN SEMIGROUPS

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Abstract

In this paper the terms, Archemedian semigroups, Strongly Archemedian semigroups are introduced. It is proved that if S is a semipseudo symmetric Archimedean semigroup then it is proved that an ideal M is maximal iff M is trivial and S has no maximal ideals if S = S^2. It is also proved that in a semipseudo symmetric semigroup S containing maximal ideals if S has no semisimple elements or S is an Archimedean semigroup then $S \neq S^2$ and $S^2 = M*$ where M* is the intersection all maximal ideals of S. It is also proved that if S is a pseudo integral semigroup then it is proved that S is strongly Archimedean, S is Archimedean, S has no proper completely prime ideals, S has no proper semiprime ideals are equivalent.

1. Introduction

The algebraic theory of semigroups was widely studied by CLIFFOD [2], [3], PETRICH [4] and LJAPIN [5]. The ideal theory in general semigroups was developed by ANJANEYULU [1]. In this paper we introduce the notions of Archemedian semigroups and Strongly Archemedian semigroups. We obtained some characterizations of Archemedian semigroups and Strongly Archemedian semigroups.

2. Preliminaries

Definition 2 .1 : A system S = (S, .), where S is a nonempty set and . is an associative binary operation on S, is called a semigroup.

Definition 2.2 : A semigroup S is said to be commutative provided $ab = ba$ for all $a, b \in S$.

Definition 2.3: A semigroup S is said to be normal provided $aS = Sa$ for all $a \in S$.

Definition 2.4: An element a of a semigroup S is said to be a two sided identity or an identity provided $as = sa = s$ for all $s \in S$.

Definition 2.5: An element a of semigroup S is said to be two sided zero or zero of S provided $sa = as = a$ for all $s \in S$.

Definition 2.6: A nonempty subset A of a semigroup S is said to be a left ideal (right ideal) of S provided it is both a left ideal and a right ideal of S.

Definition 2.8: An ideal A of a semigroup S is said to be a proper ideal of S provided $A \neq S$.

Definition 2.9: An ideal A of a semigroup S is said to be a trivial ideal of S provided S\A is singleton.

Theorem 2.10: The nonempty intersection of any family of ideals of a semigroup S is an ideal of S.

Theorem 2.11: The union of any family of ideals of a semigroup S is an ideal of S.

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Definition 2.12 : Let S be a semigroup. The intersection of all ideals of S containing a nonempty set A is called the ideal generated by A. IT is denoted by < A>.

Definition 2.13 : An ideal A of a semigroup S is said to be a principal ideal provided A is an ideal generated by single element set. If an ideal A is generated by a, then A is denoted as <a> or J[a].

Definition 2.14 : An ideal A of a semigroup S is said to be a maximal ideal provided A is a proper ideal of S and is not properly contained in any proper ideal of S.

Definition 2.15 : An ideal A of a semigroup S is said to be a minimal ideal provided A does not contain any ideal of S properly.

Definition 2.16 : An ideal A of a semigroup S is said to be a completely prime ideal provided $x, y \in S,\ xy \in A,$ implies either $x \in A$ or $y \in A.$

Definition 2.17: An ideal A of a semigroup S is said to be prime ideal provide X, Y are ideals of S, $XY \subseteq A$ implies either $X \subseteq A$ or $Y \subseteq A.$

Theorem 2.18: Every completely prime ideal of a semigroup is prime.

Definition 2.19 : If A is an ideal of a semigroup S, then the intersection of all prime ideals containing A is called prime radical or simply radical of A and it is denoted by $\sqrt{A}$ or rad A.

Definition 2.20 : An ideal A of a semigroup S is said to be completely semiprime provided $x \in S, x^n \in A$ for some natural number n implies $x \in A.$

Theorem 2.21: The nonempty intersection of any family of completely prime ideals of a semigroup is completely semiprime.

Theorem 2.22: If S is a globally idempotent semigroup then every maximal ideal M of S is a prime ideal of S.

Definition 2.23 : An ideal A of a semigroup S is said to be semiprime provided X is an ideal of S, $X^n \subseteq A$ for some natural number n implies $x \subseteq A.$

Theorem 2.24 : An ideal Q of a semigroup S is a semiprime ideal of S iff $\sqrt{Q} = Q.$

Definition 2.25 : An ideal A of a semigroup S is said to be pseudo symmetric provided x, $y \in S, xy \in A$ implies $xsy \in A$ for all $s \in S.$

Theorem 2.26: Let A be an ideal of a semigroup S. Then A is completely prime iff A is prime and pseudo symmetric.

Definition 2.27 : A semigroup S is said to be pseudo symmetric provided every ideal in S is a pseudo symmetric ideal.

Definition 2.28 : An ideal A in a semigroup S is said to be semipseudo symmetric provided for any natural number $n, x \in S, x^n \in A, \Rightarrow\ < x^n > \subseteq A.$

Theorem 2.29 : Every pseudo symmetric ideal of a semigroup is a semipseudo symmetric ideal.

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Theroem 2.30 : If A is an ideal of a semigroup S, then the following are equivalent.

(1) A is completey prime

(2) A is prime and pseudo symmetric

(3) A is prime and semipseudo symmetric.

Theorem 2.31 : Let A be a semipseudo symmetric ideal of a semigroup S. Then the following are equivalent.

1) A_1 = The intersection of all completely prime ideals of S containing A.

2) A_1^1 = The intersection of all minimal completely prime ideals of S containing A.

3) A_1^{11} = The minimal completely semiprime ideal of S relative to containing A.

4) $A_2 = \left\{x \in S : x^n \in A \text{ for some natural number } n\right\}$

5) A_3 = The intersection of all prime ideals of S containing A.

6) A_2^1 = The intersection of all minimal prime ideals of S containing A.

7) A_1^{11} = The minimal semiprime ideal of S relative to containing A.

8) $A_4 = \{x \in S :< x >^n \subseteq A \text{ for some natural number } n\}$

Corollary 2.32 : An ideal Q of a semigroup S is a semiprime ideal iff Q is the intersection of all prime ideals of S contains Q.

3. Archimedian Semigroup

Definition 3.1 : A semigroup S is said to be an archimedian semigroup provided for any $a,b \in S$ there exists a natural number n such that $a^n \in SbS$.

Definition 3.2 : A semigroup S is said to be a strongly archimedean semigroup provided for any $a,b \in S$, there is a natural number n such that $< a >^n \subseteq < b >$.

Theorem 3.3: Every strongly Archimedean semigroup is an Archimedean semigroup.

Proof: Suppose that S is strongly Archimedean semigroup. Let $a,b \in S$. Since S is strongly Archimedean semigroup, there is a natural number n such that $< a >^n \subseteq < b >$.

Now $a^n \in < a >^n \subseteq < b > \subseteq SbS$. Therefore, S is an Archimedean semigroup.

Theorem 3.4 : If S is a semipseudo symmetric semigroup, then the following are true.

1) $T = \{a \in S : \sqrt{< a >} \neq S\}$ is either empty or a completely prime ideal.

2) S\T is either empty or an Archimedean subsemigroup of S.

Proof : (1) If T is an empty set, then there is nothing to prove. If T is nonempty, then clearly T is an ideal of S. Let $a,b \in S$ and $ab \in T$. Suppose if possible $a \notin T, b \notin T$. Then $\sqrt{< a >} = S$ and $\sqrt{< b >} = S$.

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Since $ab \in T$, then $\sqrt{<ab>} \neq S$. Now $S = \sqrt{<a>} \cap \sqrt{<b>} = \sqrt{<ab>} \neq S$.

It is a contradiction. Thus $a \in T$ or $b \in T$. $\therefore$ T is a completely prime ideal.

(2) Since T is a completely prime ideal, S\T is either empty or a subsemigroup of S.

Let $a, b \in S \setminus T$. Then $\sqrt{<a>} = \sqrt{<b>} = S$. Now $b \in \sqrt{<a>}$, by the theorem 2.31.

$b^n \in <a>$ for some $n \in N$. So $b^{n-2} \in SaS \Rightarrow$ $b^{n-2} = s$ at for some $s, t \in S$.

If either s or $t \in T$, then $b^{n-2} \in T$ and hence $b \in T$. IT is a contradiction. Hence $s, t \in S \setminus T$.

Now $b^{n+2} = s$ at $\in (S \setminus T)a(S \setminus T)$. Hence S\T is an Archimedean subsemigroup of S.

Theorem 3.5 : If S is a semipseudo symmetric semigroup, then the following are equivalent.

1) S is a strongly Archimedean semigroup.
2) S is an Archimedean semigroup.
3) S has no proper completely prime ideals.
4) S haws no proper completely semiprime ideals.
5) S has no proper prime ideals.
6) S has no proper semi prime ideals.

Proof : (1) $\Rightarrow$ (2) : Suppose that S is a strongly Archimedean semigroup. By theorem 3.3, S is an Archimedean semigroup.

(2) $\Rightarrow$ (3): Suppose that S is an Archimedean semigroup. Let P be any completely prime ideal of S.

Let $a \in S, b \in P$. Since S is an Archimedean semigroup, there exists a natural number n such that $a^n \in SbS \subseteq P \Rightarrow$ $a^n \in P \Rightarrow$ $a \in P$.

$\therefore$ $S \subseteq P$ Clearly $P \subseteq S$. Thus P = S.

$\therefore$ S has no proper completely prime ideals.

By theorem 2.21, corollary 2.26, and theorem 2.30 ; (3), (4), (5) and (6) are equivalent.

(5) $\Rightarrow$ (1) : S has no proper prime ideals. Let $a, b \in S$. Since S has no proper prime ideals, $\sqrt{<b>} = S$. Now $a \in S = \sqrt{<b>} \Rightarrow$ $a^n \in <b>$ for some natural number n. Since S is a semipseudo symmetric semigroup, $<b>$ is a semipseudo symmetric ideal and hence $a^n \in <b> \Rightarrow$ $<b>^n \subseteq <b>$. Thus S is a strongly Archimedean semigroup. Hence the given conditions are equivalent.

Corollary 3.6 : A commutative semigroup S is Archimedean iff S has no proper prime ideals.

Proof : Since S is a commutative semigroup, S is a semipseudo symmetric semigroup. By theorem 3.5, S is Archimedean iff S has no proper prime ideals.

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Theorem 3.7: If M is a nontrivial maximal ideal of a semipseudo symmetric semigroup S then M is prime.

Proof: Suppose if possible M is not prime. Then there exists $a,b \in S \setminus M$ such that $<a><b> \subseteq M$. Now for any $x \in S \setminus M$, we have $S = M \cup <b> = M \cup <x>$. Since $b,x \in S \setminus M$, we have $b \in <x>$ and $x \in <b>$. So $<b> = <x>$. Therefore, $<b>^2 \subseteq M$. If $a \neq b$, then $a = sbt$ for some $s,l \in S^1$ and one of them is not an empty symbol. So $a \in <s><b><f>$. If either $s \in M$ or $t \in M$ then $a \in M$. It is a contradiction.

If $s \notin M$ and $t \notin M$, then $<s><b> \subseteq <b>^2 \subseteq M$.

$\therefore a \in <s><b><t> \subseteq M$. $\therefore a \in M$. It is a contraction. Thus $a = b$ and hence M is trivial, which is not true. So M is prime.

Definition 3.8 : An element a of a semigroup S is said to be semisimple provided

$a \in <a>^2$, that is $<a>^2 = <a>$

Theorem 3.9 : If S is a semipseudosymmetric semigroup and contains a nontrivial maximal ideal then S contains semisimple elements.

Proof: Let M be a nontrivial maximal ideal of S. By theorem 3.7, M is prime.

Let $a \in S \setminus M$. Then $<a> \not\subseteq M$. Since M is maximal, $M \cup <a> = S$.

If $<a>^2 \subseteq M$ then $<a> \subseteq M$ which is not true. So $<a>^2 \not\subseteq M$.

Since M is maximal, $M \cup <a>^2 = S$. Now $M \cup <a> = M \cup <a>^2$.

Therefore, $a \in <a>^2$ and hence a is semisimple.

Theorem 3.10 : Let S be a semipseudo symmetric Archimedean semigroup. Then an ideal M is maximal iff it is trivial, and S has no maximal ideals if $S = S^2$.

Proof: If M is trivial, then clearly M is maximal ideal. Conversely suppose that M is maximal. Suppose if possible M is nontrivial. By theorem 3.7, M is prime. Since S is an Archimedean Semigroup, by theorem 3.5, S has no prime ideals. It is a contradiction. So M is trivial. If $S = S^2$, then by theorem 2.22 every maximal ideal is prime and hence S has no maximal ideals.

Theorem 3.11 : Let S be a semipseudo symmetric semigroup containing maximal ideals. If either S has no semisimple elements or S is an Archimedean semigroup, then $S \neq S^2$ and $S^2 = M*$ where $M*$ denotes the intersection of all maximal ideals.

Proof: Suppose that S has no semisimple elements. Then by corollary 3.9, every maximal ideal is trivial. So if M is maximal, then $S = M \cup \{a\}, a \notin M$. Suppose $a \in S^2$.

Then $a \in S^2 \Rightarrow a = bc$ for some $b,c \in S$.

If $b \neq a$ then $b \in M$ and hence $bc \in M$ (Since M is Maximal) $\Rightarrow a \in M$. It is a contradiction.

$\therefore b = a$. Similarly we can prove $c = a. \therefore a = bc = a^2. \therefore a$ is semisimple.

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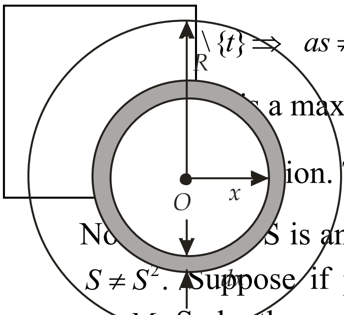
It is a contradiction. $\therefore\ a \notin S^2, \therefore\ S \neq S^2$ and $S^2 \subseteq M$. Let $t \in M*$ and $t \notin S^2$.



$\setminus \{t\} \Rightarrow\ as \neq 1, sa \neq t$ for all $s \in S \Rightarrow\ as, sa \in S\setminus\{t\} \Rightarrow\ S\setminus\{t\}$ is an ideal.

a maximal ideal. Hence $t \in S\setminus\{t\}$.

ion. Therefore, $M* \subseteq S^2$. Hence $S^2 = M*$.

No S is an Archimedean semigroup. Since S has maximal ideals, by theorem 3.10, $S \neq S^2$. Suppose if possible $x \in S^2 \setminus M*$. Then there exists a maximal ideal M, such that $x \in M$. So by theorem 3.10, $M = S\setminus\{x\}$.

Since $x \in S^2, x = yz$ for some $y, z \in S$. If either y or $z \in M$, then $x \in M$. It is a contradiction. Therefore, $y = z = x$ and hence $x = x^2$. Let $a, b \in S$, $a, b \in M$. Suppose if possible.

$a \notin M, b \notin M$. Then $a = x, b = x$. Therefore, $ab = xx = x \notin M$. It is a contradiction. Thus M is prime by theorem 3.5, S has no proper prime ideals. It is a contradiction.

Thus $S^2 \subseteq M*$. As above, we can show that $M* \subseteq S^2$. Therefore, $S^2 = M*$.

Corollary 3.12 : Let S be a commutative semigroup containing maximal ideals. If either S has no idempotents or S is an Archimedean semigroup, then $S \neq S^2$ and $S^2 = M^{*}$.

Proof : Suppose that S has no idempotents. If S contains a semisimple element a then a is regular.

Hence there exists and element $x \in S$ such that $axa = a$. Now ax is an idempotent in S. It is a contradiction. So S has no semisimple elements. Then by theorem 3.11, we have $S \neq S^2$ and $S^2 = M*$.

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