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# Investigating Pre A* Algebras as an Emerging Paradigm

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**ABSTRACT:**

This paper explores Boolean algebra postulate systems, aiming to derive a minimal set of axioms. It delves into the algebraic structure of Pre A*-algebras and establishes the congruence relation on them. Defining a ternary operation as a conditional statement, the paper explores its properties. Pre A* algebras are obtained from Boolean Algebra B, and it is proven that if A is a Pre A*-algebra and x is in A, then Ax is a Pre A*-algebra, isomorphic to a quotient algebra of A.

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**KEY WORDS:** Pre A*-Algebra, Stone type representation, Boolean algebra.

## § 0. INTRODUCTION:

In a drafted paper [6], The Equational theory of Disjoint Alternatives, around 1989, E.G.Maines introduced the concept of Ada, $(A,\wedge,\vee,(-)',(-)_\pi,0,1,2)$ which however differs from the definition of the Ada [7]. While the Ada of the earlier draft seems to be based on extending the If-Then-Else concept more on the basis of Boolean algebras, the latter concept is based on C-algebras $(A,\wedge,\vee,(-)^{\sim})$ introduced by Fernando Guzman and Craig C. Squir [3].

In 1994, P.Koteswara Rao [5] firstly introduced the concept of A*-algebra $(A,\wedge,\vee,*,(-)^{\sim}(-)_\pi,0,1,2)$ and studied the equivalence with Ada [6], C-algebra [3], and Ada [7] and its connection with 3-

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ring, Stone type representation and introduced the concept of A*-clone and the If-Then-Else structure over A*-algebra and ideal of A*-algebra.

In 2000, J.Venkateswara Rao [8] introduced the concept Pre A*-algebra $(A,\wedge,\vee,(-)^{\sim})$ analogous to C-algebra as a reduct of A*- algebra.

## § 1. BOOLEAN ALGEBRA:

**1.1. Definition:** A Boolean algebra is algebra $(B,\wedge,\vee,(-)',0,1)$ with two binary operations, one unary operation (called complementation), and two nullary operations which satisfies:

(1) $(B,\wedge,\vee)$ is a distributive lattice.

(2) $x\wedge 0=0,\ \ x\vee 1=1$ for all $x\in B$.

(3) $x\wedge x'=0,\ \ x\vee x'=1$ for all $x\in B$.

We can easily verify that $x''=x,\ (x\vee y)'=x'\wedge y',\ (x\wedge y)'=x'\vee y'$ for all $x,y\in B$.

**1.2.Note:**Alternative systems of postulates of Boolean Algebras were intensively studied during the decades 1900-1940. E.V.Huntingtion wrote an influential early paper [4] on this subject. No attempt will be made here to survey the extensive literature on such postulate systems. We present here Huntington's postulates and a new set of postulates of our own for Boolean algebra.

**1.3. Huntington's Theorem [1]**: Let B has one binary operation $\vee$ and one unary operation $(-)'$ and define

(i) $a\wedge b=(a'\vee b')',\forall a,b\in B$.

Suppose for all a, b, c $\in$B,

(ii) $a\vee b= b\vee a$, (iii) $a\vee(b\vee c)=(a\vee b)\vee c$ and

(iv) $(a\wedge b)\vee(a\wedge b')=a$. Then B is a Boolean algebra.

**1.4. Theorem [8]:** Let B has one binary operation $\wedge$ and one unary operation $(-)'$ and define

(i) $a\vee b=(a'\wedge b')',\ \ \forall a,b\in B$.

Suppose for all a, b, c $\in$B,

(ii) $a\vee b= b\vee a$, (iii) $a\vee(b\vee c)=(a\vee b)\vee c$ and

(iv) $(a\wedge b)\vee(a\wedge b')=a$. Then B is a Boolean algebra.

## § 2.Pre A* Algebra:

**2.1. Definition:**

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An algebra (A, $\wedge$, $\vee$, $(-)^{\sim}$) satisfying

(a) $x^{\sim\sim} = x$ , $\forall$ x$\in$A,   (b) $x \wedge x = x$ , $\forall$ x$\in$A,

(c) $x \wedge y = y \wedge x$ ,  $\forall$ x, y $\in$A,

(d) $(x \wedge y)^{\sim} = x^{\sim} \vee y^{\sim}$ , $\forall$ x, y $\in$A,

(e) $x \wedge (y \wedge z) = (x \wedge y) \wedge z$ ;  $\forall x, y, z \in A$,

(f) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ ; $\forall x, y, z \in A$,

(g) $x \wedge y = x \wedge (x^{\sim} \vee y)$ for all x, y, z $\in$ A,

is called a Pre A*-algebra

**2.2. Eample:**

**3** = {0, 1, 2} with $\wedge$, $\vee$, $(-)^{\sim}$ defined below is a Pre A*-algebra.

| $\wedge$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 0 | 2 |
| 1 | 0 | 1 | 2 |
| 2 | 2 | 2 | 2 |

| $\vee$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0 | 1 | 2 |
| 1 | 1 | 1 | 2 |
| 2 | 2 | 2 | 2 |

| x | $x^{\sim}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |
| 2 | 2 |

**2.3. Note:**The elements 0, 1, 2 in the above example satisfy the following laws:

(a) $2^{\sim} = 2$                (b) $1 \wedge x = x$ for all x $\in$ **3**

(c) $0 \vee x = x$ for all x $\in$ **3**   (d) $2 \wedge x = 2 \vee x = 2$ for all x $\in$ **3**.

**2.4. Example:** **2** = {0, 1} with  $\wedge$, $\vee$, $(-)^{\sim}$ defined below is a Pre A*-algebra.

| $\wedge$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 0 |
| 1 | 0 | 1 |

| $\vee$ | 0 | 1 |
|---|---|---|
| 0 | 0 | 1 |
| 1 | 1 | 1 |

| x | $x^{\sim}$ |
|---|---|
| 0 | 1 |
| 1 | 0 |

**2.5. Note:** Actually $(2, \vee, \wedge, (-)^{\sim})$ is a Boolean algebra. So every Boolean algebra is a Pre A* algebra.

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**2.6. Definition:** Let A be a Pre A*-algebra. An element x $\in$A is called central element of A if $x \vee x^{\sim} = 1$ and the set

{ x $\in$ A/ $x \vee x^{\sim} = 1$ } of all central elements of A is called the centre of A and it is denoted by B(A).The set B(A) is a Boolean algebra with $\vee, \wedge, (-)^{\sim}$ .

**2.7. Lemma:** Every Pre A*-algebra satisfies the following laws.

(a) $x \vee (x^{\sim} \wedge x) = x$ (b) $(x \vee x^{\sim}) \wedge y = (x \wedge y) \vee (x^{\sim} \wedge y)$

(c) $(x \vee x^{\sim}) \wedge x = x$ (d) $x \wedge x^{\sim} \wedge y = x \wedge x^{\sim}$

**Proof:** (a)

We have x $\wedge$ y = x $\wedge$ (x$^{\sim}$ $\vee$ y) (By 2.1 (g))

$\Rightarrow x \wedge x = x \wedge (x^{\sim} \vee x)$

$\Rightarrow (x \wedge x)^{\sim} = (x \wedge (x^{\sim} \vee x))^{\sim}$

$\Rightarrow x^{\sim} = x^{\sim} \vee (x^{\sim} \vee x)^{\sim}$ (By 2.1 (b))

$\Rightarrow x^{\sim} = x^{\sim} \vee (x \wedge x^{\sim})$

$\Rightarrow x = x \vee (x^{\sim} \wedge x)$

(b) Use 2.1(f) and 2.1(c)

(c) $(x \vee x^{\sim}) \wedge x = (x \wedge x) \vee (x^{\sim} \wedge x)$ (By 2.7 (b))

$= x \vee (x^{\sim} \wedge x)$ (By 2.1 (b))

= x (By 2.7 (a))

(d) Can be verified routinely.

**2.8. Lemma:** Let A be a Pre A*algebra with 1, 0 and let

x, y $\in$A

(a) If $x \vee y = 0$, then x = 0

(b) If $x \vee y = 1$, then $x \vee x^{\sim} = 1$

**Proof:** (a) $x = x \vee 0$

$= x \vee x \vee y$ $(x \vee y = 0)$

$= x \vee y$ = 0 $(2.1b)^{\sim}$

( b) 1 = $x \vee y$

= $x \vee (x^{\sim} \wedge y)$ $(2.1g)^{\sim}$

= $(x \vee x^{\sim}) \wedge (x \vee y) = (x \vee x^{\sim}) \wedge 1 = (x \vee x^{\sim})$

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**2.9. Definition:** A relation θ on a Pre –A* algebra (A, ∧, ∨, (-)˜) is called congruence relation if

(i) θ is an equivalence relation

(ii ) θ is closed under ∧, ∨, (-)˜ .

**2.10. Lemma:** Let (A, ∧, ∨, (-)˜) be a Pre A*-algebra and let $a \in A$.Then the relation

$\theta_a = \{(x, y) \in A \times A / a \wedge x = a \wedge y\}$ is

(i) a congruence relation (ii) $\theta_a \cap \theta_{a'} = \theta_{a \vee a'}$

we will write $x\theta_a y$ to indicate $(x, y) \in \theta_a$

**2.11. Definition:** Let A be a Pre -A* algebra. If

x, p, q $\in A$, define the ternary operation $\Gamma_x(p,q) = (x \wedge p) \vee (x^{\sim} \wedge q)$ ( $\Gamma_x(p,q)$ should be viewed as conditional " if x, then p, else q").

**2.12. Lemma:** Every Pre-A* algebra with the indicated constants satisfies the following laws.

( i ) $\Gamma_2(p,q) = 2$ (ii) $\Gamma_x(2,2) = 2$

(iii) $\Gamma_1(p,q) = p$ (iv) $\Gamma_0(p,q) = q$ (v) $\Gamma_x(1,0) = x$

Proof: By inspection.

**Definition:** Let A be a Pre A*- algebra and $x \in A$.Define $\psi_x = \{(p,q) \in A \times A / \Gamma_x(p,q) = p\}$

**Lemma:** Let A be a Pre A*-algebra and $x \in A$. Then

(i) $\psi_x \subseteq \theta_{x'}$

(ii ) $\psi_x$ is transitive but it is neither reflexive nor symmetric.

**Theorem:** Let A be a Pre A* algebra with 1 and x∈ B(A) then

(i) $\psi_x = \theta_{x'}$ (ii) $\psi_x$ is congruence relation on A

**2.13. Lemma:** Every Pre-A* algebra satisfies the laws:

(i) $\Gamma_x(p,q)^{\sim} = \Gamma_x(p^{\sim}, q^{\sim})$

(ii) $\Gamma_x(p,q) \wedge r = \Gamma_x(p \wedge r, q \wedge r)$

(iii) $\Gamma_x(p,q) \vee r = \Gamma_x(p \vee r, q \vee r)$

(iv) $\Gamma_x(\Gamma_y(p,q), \Gamma_y(r,s)) = \Gamma_y(\Gamma_x(p,r), \Gamma_x(q,s))$

**2.14. Definition:** Let $(A_1, \vee, \wedge, (-)^{\sim})$ and $(A_2, \vee, \wedge, (-)^{\sim})$ be a two Pre A*- algebras. A mapping $f : A_1 \to A_2$ is called an Pre A*-homomorphism if

(i) $f(a \wedge b) = f(a) \wedge f(b)$ (ii) $f(a \vee b) = f(a) \vee f(b)$ (iii) $f(a^{\sim}) = (f(a))^{\sim}$ .

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If in addition, f is bijective, then f is called an Pre A*-isomorphism, and $A_1, A_2$ are said to be isomorphic, denoted in symbols $A_1 \cong A_2$.

**2.15. Lemma:** Let A be a Pre A*-algebra with 1, 0. Suppose that for every x ∈ A – {0, 1}, x ∨ x$^{\sim}$ ≠ 1. Define f : A → {0, 1, 2} by f(1) = 1, f(0) = 0 and f(x) = 2 if x ≠ 0, 1. Then f is a Pre A*-algebra homomorphism.

**§ 3 Generating Pre A* algebras:**

In this section we generated Pre A*- algebras A(B)=$\{(a_1,a_2)/\,a_1,a_2 \in B \text{ and } a_1 \wedge a_2 = 0\}$ and $A_B = B\times B/\approx\ = \{\langle a,b\rangle / (a,b) \in B\times B\}$ from Boolean algebra where $\langle a,b\rangle = \{(c,d)\in B\times B/(a,b)\approx(c,d)\}$, the equivalence class containing (a,b), ≈defined on $B\times B$ as $(a,b)\approx(c,d)$ if and only if $a=c$ and $a'b=c'd$ and also we proved that $A_B \cong A(B)$. First we prove the following

**3.1. Theorem:** Let $(B,\wedge,\vee,(-)',0,1)$ be a Boolean Algebra. Then A(B)=$\{(a_1,a_2)/\,a_1,a_2 \in B \text{ and } a_1 \wedge a_2 = 0\}$ becomes a Pre A* algebra with 1= (1,0) , 0 = (0,1), 2=(0,0) and ∀ a , b ∈A(B),

(i) a ∧ b = $(a_1b_1, a_1b_2 + a_2b_1 + a_2b_2)$ where juxta position , +, $(-)'$ respectively ∧ , ∨ , $(-)'$ in Boolean algebra B (ii) a ∨ b = $(a_1b_1 + a_1b_2 + a_2b_1, a_2b_2)$ and (iii) $a^{\sim} = (a_2, a_1)$

**3.2. Theorem:** Suppose $(B,\vee,\wedge,(-)',0,1)$ is Boolean algebra. Define ≈ on $B\times B$ as $(a,b)\approx(c,d)$ if and only if $a=c$ and $a'b=c'd$ .Then

(i) ≈ is an equivalence relation on $B\times B$ ; $\langle a,b\rangle = \{(c,d)\in B\times B/(a,b)\approx(c,d)\}$, the equivalence class containing (a,b).Let $A_B = B\times B/\approx\ = \{\langle a,b\rangle / (a,b)\in B\times B\}$.

(ii)For every $\langle a,b\rangle \in A_B$ there exists e,f $\in B$

and ef=0 such that (e,f) $\in \langle a,b\rangle$ and (e,f) is unique.

(iii) Define, $\vee,\wedge,(-)^{\sim}$ on $A_B$ as follows:

Assume that $A_B = \{\langle a,b\rangle / a,b \in B, ab=0\}$.

(a) $\langle a,b\rangle \wedge \langle c,d\rangle = \langle ac, ad+bc+bd\rangle$ where juxta position, +, respectively $\wedge,\vee$ in Boolean algebra B.

(b) $\langle a,b\rangle \vee \langle c,d\rangle = \langle ac+ad+bc, bd\rangle$

(c ) $\langle a,b\rangle^{\sim} = \langle b,a\rangle$ .Then $(A_B,\vee,\wedge,(-)^{\sim})$ is a pre A* algebra.

(Note that in Pre A*-algebra $(A_B,\vee,\wedge,(-)^{\sim})$ , $1 = <1,0>$, $0 = <0,1>$, $2 = <0,0>$).

**4. The Pre A*-algebra $A_x$ :**

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Recall that for every Boolean algebra B and $a \in B$ the set(a] ={ $x \in B$ / $x \leq a$} ([a) ={ $x \in B$ / $x \leq a$}) is a Boolean algebra under the induced operations $\wedge$ ,$\vee$ where the complementation is defined by $x^* = a \wedge x'$ ($x = a \vee x'$ )

In this section we prove that if A is a pre A*-algebra and $x \in A$, then $A_x = \{s \in A / s \leq x\}$ is a Pre A*-algebra under the induced operations and $A_x$ is isomorphic to a quotient algebra of A.

**4.1 Theorem:** Let A be a Pre A* algebra, $x \in A$, and $A_x = \{s \in A / s \leq x\}$.Then $< A, \wedge, \vee, * >$ is Pre A* algebra with 1 where $\wedge, \vee$ are the operations in A restricted to $A_x$ , s* is defined by $x \wedge s^{\sim}$ , the relation defined on Pre A* algebra A by $s \leq x$ if $s \wedge x = x \wedge s = x$

**Proof :** If $s \in A_x$ ,then

$x \wedge s^* = x \wedge (x \wedge s^{\sim}) = (x \wedge x) \wedge s^{\sim} = x \wedge s^{\sim}$ =s*.

So that $s^* \in A_x$ and

s**=(s*)*= $(x \wedge s^{\sim})^* = (x \wedge s^{\sim})^{*} = x \wedge (x \wedge s^{\sim})^{\sim}$

$= x \wedge (x^{\sim} \vee s) = x \wedge s = s$

Now, for $s, t \in A_x$, $(s \wedge t)^* = x \wedge (s \wedge t)^{\sim} = x \wedge (s^{\sim} \vee t^{\sim})$

$= (x \wedge s^{\sim}) \vee (x \wedge t^{\sim})$ =s* v t*

For $s, t \in A_x$

$s \wedge (s^* \vee t) = s \wedge ((x \wedge s^{\sim}) \vee t) = s \wedge (x \wedge s^{\sim}) \vee (s \wedge t)$

$= s \wedge (s^{\sim} \wedge x) \vee (s \wedge t) = (s \wedge s^{\sim}) \vee (s \wedge t)$

$= s \wedge (s^{\sim} \vee t) = s \wedge t$ (since $s, t \in A_x$)

The remaining properties hold in $A_x$ since they hold in A. Hence $< A, \wedge, \vee, * >$ is a Pre A* algebra . Observe that $A_x$ is not a sub-algebra of A because the operation * is not the restriction *of* ~ to $A_x$ .

**4.2 Theorem:** Let A be a Pre A*- algebra. Then the following hold:

(i) $A_x = \{x \wedge s / s \in A\}$

(ii) $A_x = A_y$ if and only if x = y (iii) $A_x \cap A_y \subseteq A_{x \wedge y}$

(iv) $A_x \cap A_{x'} = A_{x \wedge x'}$ (v) $(A_x)_{x \wedge y} = A_{x \wedge y}$

**Proof:**

(i), (ii) and (iii) can be proved routinely.

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For (iv) , Let $s \in A_{x \wedge x'}$ , then $(x \wedge x^{\sim}) \wedge s = s$

Now $x \wedge s = x \wedge (x \wedge x^{\sim} \wedge s) = x \wedge x^{\sim} \wedge s = s$

Again $x^{\sim} \wedge s = x^{\sim} \wedge (x \wedge x^{\sim} \wedge s) = x \wedge x^{\sim} \wedge s = s$

For (v) , $(A_x)_{x \wedge y} = \{x \wedge y \wedge t / t \in A_x\}$

$= \{x \wedge y \wedge x \wedge s / s \in A\}$

$= \{x \wedge y \wedge s / s \in A\} = A_{x \wedge y}$

**4.3 Lemma:** Let $f : A_1 \to A_2$ be Pre A* - algebra homomorphism where $A_1$, $A_2$ are Pre A* algebras with $1_1$and $1_2$ . Then (i) If $A_1$ has the element 2, then f(2) is the element of $A_2$

(ii) If $a \in B(A_1)$ , then $f(a) \in B(A_2)$

**4.4 Theorem:** Let A be a Pre A*-algebra with 1 and $x \in A$, then the mapping $\alpha_x : A \to A_x$ defined by $\alpha_x(s) = x \wedge s$ for all $s \in A$ is a homomorphism of A onto $A_x$ with kernel $\theta_x$ and hence $A / \theta_x \cong A_x$

Proof: For $s \in A$, $x \wedge s \le x$ and hence $x \wedge s \in A_x$ .

Let s, t $\in$ A, then

$\alpha_x(x \wedge s) = x \wedge s \wedge t = x \wedge s \wedge x \wedge t = \alpha_x(s) \wedge \alpha_x(s)$

$\alpha_x(s^{\sim}) = x \wedge s^{\sim} = x \wedge (x^{\sim} \vee s^{\sim}) = x \wedge (x \wedge s)^{\sim}$

$= (x \wedge s)^* = (\alpha_x(s))^*$

We can prove that $\alpha_x(s \vee t) = \alpha_x(s) \vee \alpha_x(t)$. Hence $\alpha_x$ is a Pre A* homomorphism. Now $s \in A_x$, we have $\alpha_x(s) = s$. Therefore $\alpha_x$ is onto homomorphism. Hence by the fundamental theorem of homomorphism $A / \ker \alpha_x \cong A_x$ and Ker $\alpha_x = \{(s,t) \in A \times A / \alpha_x(s) = \alpha_x(t)\}$

$= \{(s,t) \in A \times A / x \wedge s = x \wedge t\} = \theta_x$ Thus $A / \theta_x \cong A_x$

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