ON PRIME LABELING OF SOME SPECIAL GRAPHS

M. K. Karthik Chidambaram
Assistant Professor, Department of Mathematics , S. T. Hindu College, (Affiliated to Manonmaniam Sundaranar University, Tirunelveli) Nagercoil, Tamil Nadu, India
karthikmat5@gmail.com

Abstract
Let G=(V,E) be a graph. A bijection f:V→{1,2,…,|V|} is called a prime labeling if for each edge e=uv in E, we have gcd{f(u),f(v)}=1. A graph that admits a prime labeling is called a prime graph. In this paper, we prove that the Jewel graph J_n , Jelly fish graph $J_{m,n}$, the graph $S_{m,n}$, the Headwood graph ,the Butterfly graph, the Dumbbell graph are prime cordial graphs. We also characterize the n-sided Prism graph that are prime cordial.

Keywords– Prime graph, Jewel graph , Jelly fish graph , Dumbbell graph
AMS Subject classification : 05C10

1. INTRODUCTION

Let G=(V,E) be a simple undirected graph. A bijection f:V→{1,2,…,|V|}is called a prime labeling if for each edge e = uv in E, we have gcd (f(u),f(v)) = 1. A graph that admits a prime labeling is called a prime graph. The notion of prime labeling was introduced by Roger Entringer and was discussed in [4, 6, 8]. Terms not defined here are used in the sense of Harary[3].

Definition 1.1.[5] Jewel graph J_n

Jewel graph J_n is obtained from a 4-cycle with vertices x,y,u,v by joining x and y with a prime edge and also appending edges from u and v which meets at common vertices v_i $(1 \leq i \leq n)$. Prime edge in a Jewel graph is defined to be the edge joining the vertices x and y.

Definition 1.2.[5] Jelly fish graph $J_{m,n}$

Jelly fish graph $J_{m,n}$ is obtained from a cycle C_4, uxvyu by joining x and y with an edge and appending m pendent edges to u and n pendent edges to v.

Definition 1.3.[5]

The Complete bipartite graph $K_{1,n}$ is called a star. We denote by $S_{m,n}$, a star with n spokes in which each spoke in a path of length m.

Definition 1.4.[5]

Heawood graph is a cubic graph with 14 vertices and 21 edges as given in Fig.2.2 .

Definition 1.5.[5]

The graph obtained by appending m pendent edges to the common vertex shared by two even cycles of the same order n, say C_n is called the Butterfly graph $BF_{m,n}$.

Definition 1.6.[5]

The graph obtained by joining two disjoint cycles $u_1u_2u_3....u_n\ u_1$ and $v_1v_2v_3.....v_nv_1$ with an edge u_1v_1 is called Dumbbell graph Db_n .

Definition 1.7[5]

Let $G_1 = (V_1 , E_1)$ and $G_2 = (V_2 , E_2)$ be two graphs. Their Cartesian product $G_1 \times G_2$ is defined as a graph with vertex set $V_1 \times V_2$ and two vertices (u_1 , v_1) and (u_2 , v_2) are adjacent if and only if either $u_1 = u_2$ and $v_1 v_2 \in E_2$ or $u_1 u_2 \in E_1$ and $v_1 = v_2$.

Definition 1.8.[5]

An n-sided prism Pr_n is a planar graph having two faces viz., an inner face and outer face with n sides and every other face is a 4-cycle. In other words, it is $C_n \times K_2$.

We use the following theorems:

Joseph Bertrand [1] conjectured in 1845 that for each integer $n \geq 2$, there is a prime p such that $n < p < 2n$. This was named as Bertrand's postulate. It was proved by Chebyshev [7] in 1852 and a shorter but also advanced proof was given by Ramanujan [6] in 1919 .

Theorem 1.1[1] [Bertrand-Chebyshev theorem]

For each integer $n \geq 2$, there exists a prime p such that $n < p < 2n$.

2. MAIN RESULTS

Definition 2.1 [4]

Let G=(V,E) be a graph. A bijection $f:V\rightarrow\{1,2,\ldots,|V|\}$ is called a prime labeling if for each edge e = uv in E, we have $\gcd(f(u),f(v)) = 1$. A graph that admits a prime labeling is called a prime graph.

Example 2.2

The Bull graph is a prime graph. Fig. 2.1 gives a prime labeling of the Bull graph.

IJFANS INTERNATIONAL JOURNAL OF FOOD AND NUTRITIONAL SCIENCES
ISSN PRINT 2319 1775 Online 2320 7876
, UGC CARE Listed (Group -I) Journal Volume 11, Iss 11, 2022

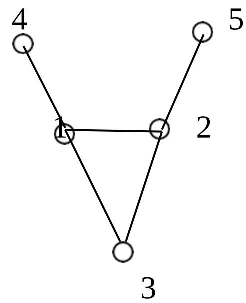

Fig. 2.1

Theorem 2.3

The Jewel graph J_n ($n \geq 0$) is a prime graph.

Proof.

Let $G \approx J_n$. The Jewel graph J_n has n+4 vertices and 2n+5 edges. If n=0, the Jewel graph has 4 vertices , u, v, x and y. Define f: $V(J_0) \rightarrow \{1,2,3,4\}$ by f(x)=1, f(y)=3, f(u)=2 and f(v)=4. Clearly f is a prime labeling.

Suppose $n \geq 1$. If n is itself a prime number choose p=n. Suppose n is not a prime. If n is even , by Theorem 1.1, there exists a prime number q_1 such that $\frac{n}{2} < q_1 < n$. Choose $p = q_1$. If n is odd, again by Theorem 1.1, there exists a prime number q_2 such that $\frac{n+1}{2} < q_2 < n+1$.

Choose $p = q_2$. Define f:V(G) $\rightarrow \{1, 2, \ldots, n+4\}$ as follows:

f(u)=1, f(x)=2, f(y)=3, f(v)=p and f(v_i) ($1 \leq i \leq n+4$) takes distinct values arbitrarily from the set of remaining integers $\{4,\ldots,p-1,p+1,\ldots,n\}$. Clearly f is a prime labeling since by choice of p, we have 2p > n for all n. □

Theorem 2.4

The Jelly fish graph $J_{m,n}$ (m, $n \geq$ o) is a prime graph.

Proof.

The Jelly fish graph has m+n+4 vertices and m+n+5 edges. Let $G \approx J_{m,n}$. Let u_i ($1 \leq i \leq m$) be the pendent vertices appended to u and v_j ($1 \leq j \leq n$) be the pendent vertices appended to v.

If m=n=0, $J_{m,n}$ is same as the Jewel graph J_0 and so prime. If not, define

f: V(G) $\rightarrow \{1, 2,\ldots,m+n+4\}$ as follows:

f(u) =1, f(x)=2, f(y)=3, f(v)=p where p is chosen as in Theorem 2.3. Let f(u_i) and f(v_j) ($1 \leq i \leq m$, $1 \leq j \leq n$) take distinct values arbitrarily from the set of remaining integers $\{4,\ldots,p-1 , p+1,\ldots,m+n+4\}$. As in Theorem 2.3, f is a prime labeling . □

Theorem 2.5

The graph $S_{m,n}$ is a prime graph for all m, n.

IJFANS International Journal of Food And Nutritional Sciences

Proof.

The graph $S_{m,n}$ has mn+1 vertices and mn edges. Let u be the root vertex . Let the vertices of the n spokes, each of length m be labeled as v_{ij} $(1 \le i \le n,\ 1 \le j \le m)$ with vertices of i^{th} spoke having labels $v_{i1}, v_{i2}, \ldots, v_{im}$ $(1 \le i \le n)$. Let $G \approx S_{m,n}$.

Define $f : V(G) \to \{1, 2, \ldots, mn+1\}$ as follows:

$f(u)=1$; For i=1, $f(v_{1j})$ $((1 \le j \le m)$ are labeled with 2, 3, 4,…, m+1 consecutively. In general the vertices in the i^{th} spoke $v_{i1}, v_{i2}, \ldots, v_{im}$ $(1 \le i \le n)$ are labeled with (i-1)m+2, (i-1)m+3,…, im+1. As the labels of the two vertices incident with an edge are either consecutive or one label is 1, all such labels are relatively prime. So f is a prime labeling. □

Theorem 2.6

The Heawood graph is a prime graph.

Proof.

Let the vertices of the Heawood graph G be labeled as in Fig. 2.2.

Define f: $V(G) \to \{1, 2, \ldots, 14\}$ as follows:

$f(v_i)$ =I for $1 \le i \le 8$, $f(v_9) = 13$, $f(10) = 12$, $f(v_{11}) = 11$, $f(v_{12}) = 10$, $f(v_9) = 13$,

$f(v_{13}) = 9$ and $f(v_{14}) = 14$. It is easy to verify that f is a prime labeling. □

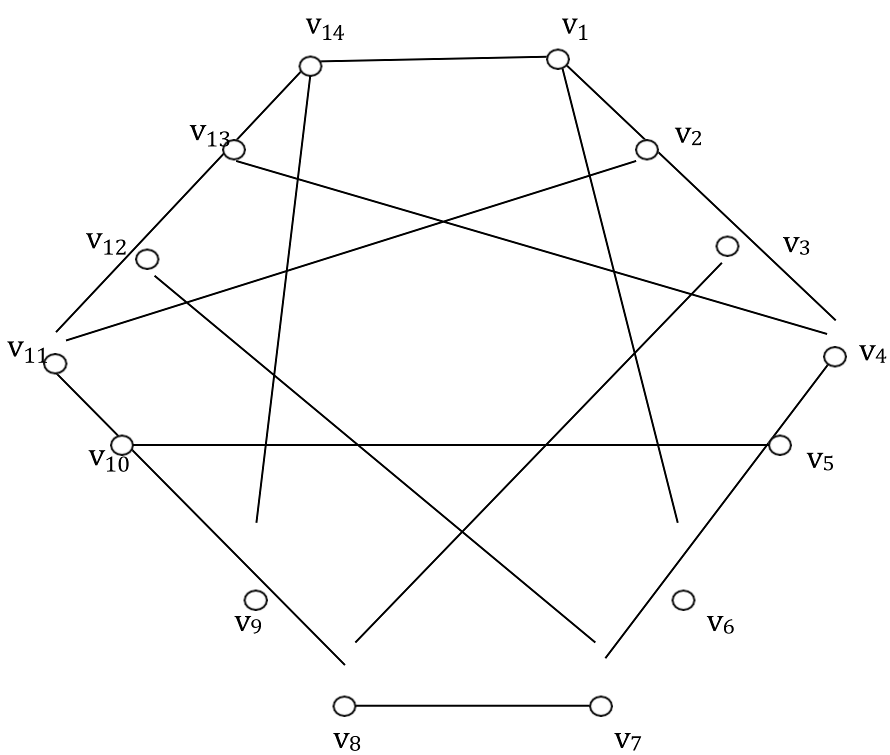

Fig. 2.2

Theorem 2.7

The Butterfly graph $BF_{m,n}$ is a prime graph.

Proof.

Let $G \approx BF_{m,n}$. G has 2n+m-1 vertices and 2n+m edges. Let the two cycles in the Butterfly graph be taken as C_1 and C_2 where vertices of C_1 are labeled as $u_1, u_2, \ldots, u_n$ and that of C_2 are labeled as $v_1, v_2, \ldots, v_n$. Note that $u_1 = v_1$. he pendent vertices are labeled as $w_1, w_2, \ldots, w_m$.

Define a function $f:V(G) \to \{1, 2, \ldots, 2n+m-1\}$ as follows:

$f(u_1) = f(v_1) = 1$

$f(u_i) = i$ for $2 \leq i \leq n$

$f(v_j) = n+j$ for $2 \leq j \leq n$

$f(w_k) = 2n-1+k$ for $1 \leq k \leq m$.

Clearly f is a prime labeling. □

Theorem 2.8

The Dumbbell graph Db_n is a prime graph.

Proof.

The graph $G \approx Db_n$ has 2n vertices and 2n+1 edges. By Theorem 1.1, there exists a prime p such that $n < p < 2n$. The vertices are denoted by notations as in Definition 1.6.

Define $f: V(G) \to \{1, 2, \ldots, 2n\}$ as follows:

$f(u_1) = 1$

$f(u_i) = I$ $(2 \leq i \leq n)$

$f(v_1) = p = n+k$ for some k.

The vertices v_j $(2 \leq j \leq n)$ are labeled consecutively with labels n+1, n+2, …, n+(k-1), n+(k+1),…, 2n. As $n+1 < p$ and p is a prime, $(n+1, p)=1$. As $n < p$, $2p > 2n$ and so $(p, 2n) = 1$. Hence f is a prime labeling. □

Theorem 2.9

The n-sided prism Pr_n is a prime graph if and only if n is even.

Proof.

Assume that Pr_n is a prime graph. It is a graph with 2n vertices and 3n edges. We claim that n is even. Suppose n is odd. Let u_i $(1 \leq i \leq n)$ and v_i $(1 \leq i \leq n)$ be the vertices of the inner cycle and the outer cycle respectively so that u_i and v_i are adjacent. At most $\left\lfloor \frac{n}{2} \right\rfloor$ vertices of the inner cycle can be labeled with even integers. Similar is the case with the vertices of the outer

ISSN PRINT 2319 1775 Online 2320 7876
 UGC CARE Listed (Group -I) Journal Volume 11, Iss 11, 2022

cycle. Thus number of vertices that can be labeled with even integers is $2\left\lfloor \frac{n}{2} \right\rfloor = 2\left(\frac{n-1}{2}\right) = n-1$ as n is odd. But there are n even integers to be used as labels. Hence the graph cannot have a prime labeling.

Conversely , assume that n is even. Let $l = \left\lfloor \frac{2n}{3} \right\rfloor$. Then there are $\left\lceil \frac{l}{2} \right\rceil$ odd labels that are multiples of 3 and $k = \left\lfloor \frac{l}{2} \right\rfloor$ even labels that are multiples of 3 . The vertices

u_1 , v_2, u_3 , v_4 etc are alternatively labeled with the even labels that are multiples of 3 until they are exhausted. So there are k+2 forbidden positions for the odd multiples of 3. The remaining even integers can be used as alternative labels in a similar way. The odd labels that are multiples of 3 are used for vertices that are not in their forbidden positions. Similar argument is repeated for all odd numbers less than n. In this way we get a prime labeling for the n-sided prism Pr_n. □

References.

[1] Bertrand Joseph, Memoire sur le nombre de valeirs que peut prendre une function quandon y permut les letters quelle remfreme, Journal de lEcole Royale Polytechnique (in French), 18(Cahler 30);123-140 (1845)

[2] Chebychev, P L , Memoire sur les nombres premiers, Journal de mathematiques pures et appliqués sene 1 (in French) , 366-390 (1852)

[3] Harary, Graph Theory, Narosa Publishing house, New-Delhi(1998)

[4] Hung-Lin Fu and Kuo-Ching Huang, On prime labeling, Discrete Mathematics 127 , 181-186 (1994)

[5] Joseph A. Gallian, A Dynamic Survey of Graph labeling, The Electronic Journal of Combinatorics, #DS6,(2022)

[6] Prajapati U. M and Gajjar S.J, Some results on prime Labeling, Open Journal of Discrete Mathematics, Vol.4, No.3, July 2014

[7] Ramanujan S, A proof of Bertrand's postulate, Journal of the Indian Mathematical Society, 11, 181-182(1919)

[8] Tout A et al., Prime labeling of graphs, National Academic Science letters, 11, 365-368 (1982)